Inefficiency

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Abstract

We introduce an ordinal model of efficiency measurement. Our primitive is a notion of efficiency that is comparative, but not cardinal or absolute. In this framework, we postulate axioms that we believe an ordinal efficiency measure should satisfy. Primary among these are choice consistency and planning consistency, which guide the measurement of efficiency in a firm with access to multiple technologies. Other axioms include scale-invariance, which says that units of measurement of commodities does not matter, strong monotonicity, which states that efficiency should decrease if the inputs and outputs remain static when the technology becomes unambiguously more efficient, and a very mild continuity condition.. These axioms characterize a family of path-based measures. By replacing the continuity condition with symmetry, which states that the names of commodities do not matter, we obtain the coefficient of resource utilization.

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1 Introduction

Since the beginning of economics as a science, economists have tried to address the fundamental question of how to measure the efficiency of economic systems. A classical answer to this problem was provided by Debreu (1951) who introduced a simple method to measure the underutilization of resources called the *coefficient of resource utilization*. Debreu's coefficient has enjoyed a very rich history in applied economics, primarily as a result of its operationalization for applied economists by Farrell (1957). See, for example, Nishimizu and Page (1982), Blomström (1986), Färe, Grosskopf, and Lovell (1985), Färe, Grosskopf, and Lovell (1994), or Färe, Grosskopf, Norris, and Zhang (1994).

Our contribution here is twofold. First, we introduce an entirely new framework for the study of efficiency measurement—the ordinal framework. This framework allows us to study efficiency measurement using an axiomatic approach without resorting to an ad-hoc cardinal benchmark. Secondly, using two properties of efficiency measures, we offer an entirely new ordinal characterization of a family of efficiency measures, which includes the coefficient as a special case. This family need not treat all commodities symmetrically, unlike the coefficient. This is the family of *path-based rules*, which we describe below. Finally, in an ordinal exercise similar to that of Christensen, Hougaard, and Keiding (1999), we also show that the coefficient emerges as the unique symmetric such rule.

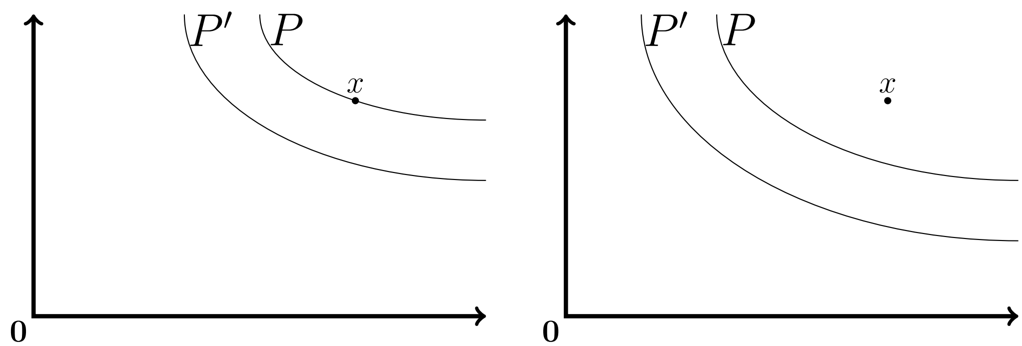
We understand efficiency measurement as the problem of comparing efficiency across different production possibility sets. Economists are routinely faced with the problem of judging how efficiently one firm performs compared to another firm. Or, a firm may be concerned with how efficiently one plant performs compared to another plant. In our conception, efficiency involves two factors: the firm's *technology*—the production possibilities available to that firm—and the choices of inputs and outputs made by that firm. Thus we study the efficiency of the chosen input/output combination *relative* to the given technology.

For a given technology, there is a set of resource bundles which could be utilized without hurting production, say P. We term P the *input set.* The resource bundle that is actually used, say x, may or may not be efficient for this input set. An *efficiency measure* is a ranking of these pairs of objects, enabling comparisons of the form: resource bundle x is more efficient for input set P than is resource bundle y for input set Q.[1] This primitive can also be found in Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999) in a cardinal form, so that a ranking is not posited, but rather is a functional representation.

To illustrate, we describe a natural property of efficiency measures that illustrates the idea of measuring efficiency production relative to a technology. Suppose that we currently produce output y with inputs x using technology P. Tomorrow, a new engineering discovery makes it possible to produce the same output using fewer inputs, resulting in technology P', so that $P \subseteq P'$. Despite the new discovery, the firm continues to produce output y with inputs x. In this case, the firm's production has become (weakly) less efficient, *even though the firm's inputs and outputs have remained exactly the same.* It is less efficient because efficiency is measured with respect to the technology. In Figure 1(a), x is efficient with respect to P in the sense that one can not

[1]Our framework therefore discusses what is typically called *input efficiency.* This modeling choice postulates an implicit independence axiom: the fact that the efficiency measure depends on the input set and not on the technology as a whole bears some relation to independence axioms found in social choice, most notably the independence of irrelevant alternatives axioms of Arrow (1963) and Nash (1950).

produce the same level of output with less of any commodity. On the other hand, x is not efficient with respect to P'. In Figure 1(b), x is inefficient under both technologies, but, for the same reason, is more efficient with respect to P than with respect to P'.



(a) x is efficient with respect to P (b) x is inefficient with respect to P

Figure 1: P and P' denote the input sets induced by their respective technologies.

The ultimate purpose of these measures is to determine which, amongst a class of possible economic units, performs the most efficiently, and is not to say *how* efficient a given unit is. To this end, efficiency measures in our model are ordinal (or comparative) and not cardinal (or absolute). A cardinal measure of efficiency can be constructed easily by applying the ordinal measure to some specific benchmark.

The coefficient of resource utilization is a cardinal efficiency measure of the underutilization of resources by a given firm. It is defined as the proportion of the given inputs required to produce the same level of output. The coefficient is illustrated in Figure 2. The point x in the figure represents the current inputs. The shaded area represents the input combinations which could be utilized without hurting production. The straight line from x to the origin contains all of the input combinations which are proportional to the original input bundle. The coefficient follows this line: it is the ratio of the length of the line segment from the origin to y (the boundary of the shaded area) to the length of the line segment from the origin to x (the current inputs).

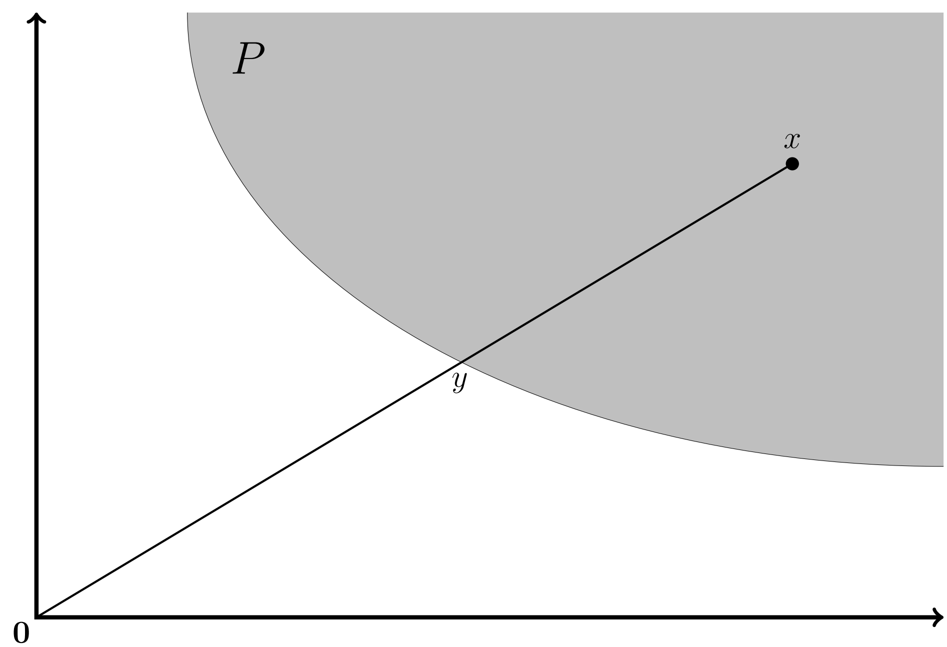


Figure 2: Coefficient of Resource Utilization: $f_c(P, x) = \frac{\|y\|}{\|x\|}$

The coefficient of resource utilization is symmetric in the sense that all inputs are treated equally. This symmetry might be undesirable to the extent that some resources might be objectively more valuable than others. This may be because some resources are more scarce than others, or, in the case of an open economy, because some resources may traded in international markets at prices which are determined exogenously. Thus if there are two factors of production, diamonds and saltwater, it may make sense for a measure to take into account that diamonds can be traded at a high price while salt water may essentially be free.

Following Luenberger (1992, 1996), Chambers, Chung, and Färe (1996) approach this problem by introducing measures which depend on other lines. These lines also pass through the current inputs, but they differ from the coefficient of resource utilization in that their gradient is determined by a numeraire which may reflect prices or intrinsic value. These measures are based on the insight that the coefficient itself measures the efficiency of production via a numeraire; specifically, the numeraire is chosen to be the observed inputs. Chambers, Chung, and Färe (1996) suggest that other numeraires are possible and may be appropriate in certain environments. (See Figure 3.)

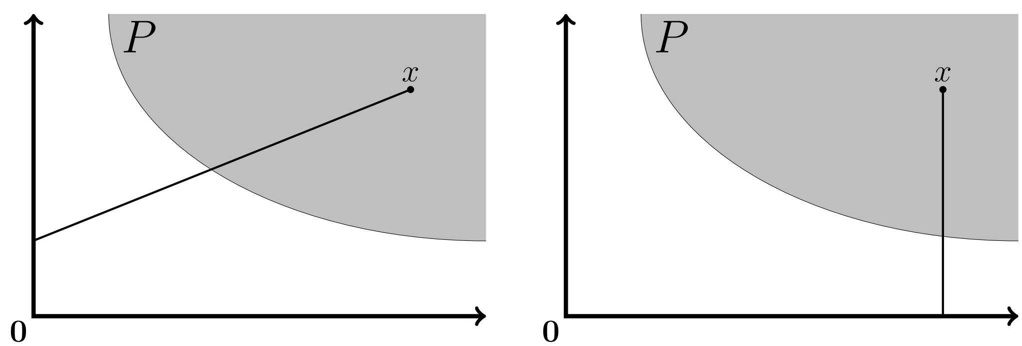


Figure 3: Chambers, Chung, and Färe (1996) suggest different numeraires.

Existing axiomatic characterizations of technical efficiency measures also presuppose the existence of a numeraire by which one can measure efficiency. (See, for example, Färe and Lovell (1978); Russell and Schworm (2009, 2010).) These measures face a weakness in that any choice of a numeraire is necessarily arbitrary.

It is not a coincidence that current approaches assume the existence of a numeraire. All existing axiomatic models of efficiency measures are defined in cardinal terms, and consequently must refer to some objectively measurable quantity. In all cases, the interpretation is measurement in terms of the commodities under consideration. This approach is analogous to measuring utility in terms of money: it makes sense to do so and is meaningful in many environments, but it severely restricts us from understanding other approaches which may be equally natural.

To remedy this problem, we introduce the first ordinal model of efficiency measurement. This ordinal approach suggests several natural axioms on efficiency measures which we believe have not been described before in an ordinal setting. We refer to these as planning and choice consistency. Versions of these axioms are found in Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999) in cardinal form.

To understand these axioms, consider a firm undertaking the following thought experiment. Suppose that there are two plants, Paradise and Quarryville, which operate under two distinct technologies, described by P and Q respectively. The firm currently has access to the plant in Paradise and produces a certain output using inputs x. (See Figure 4(a).) It would also be possible to produce the same output using inputs x in the Quarryville plant, but to do so would be judged less efficient by our ordinal efficiency measure than it would be in Paradise because of the differences in technology in the two plants.(See Figure 4(b).) Suppose that the firm has committed to producing output y tomorrow and must select its inputs today.

To understand planning consistency, assume that the firm is told that it is not clear which of the two plants will be accessible tomorrow. This results in a new technology, described by $P \cap Q$, because today it must choose inputs which would be sufficient to produce the level of output under both of the technologies. (See Figure 4(c).) Planning consistency requires the following: given that it was relatively efficient to produce the output with inputs x under technology P, it should not be more efficient to produce the output with inputs

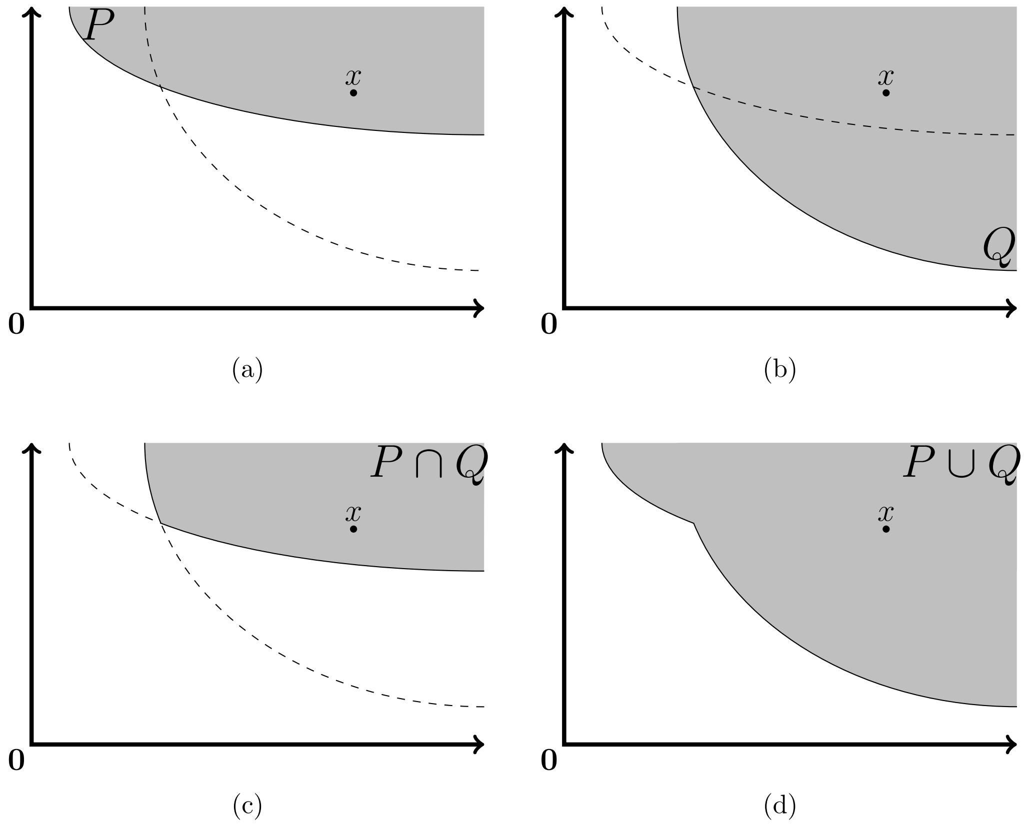


Figure 4: The gray areas represent the input sets corresponding to the various technologies.

x under technology $P \cap Q$.

To understand choice consistency, assume instead that the firm is told that it will be able to choose to produce in either of the two plants tomorrow (but not both). This results in a new technology, described by $P \cup Q$, because the firm may choose inputs that would be sufficient under either of the two technologies. (See Figure 4(d).) Choice consistency requires the following: given that it was relatively inefficient to produce the output with x under technology Q, it should not be less efficient to produce the output with inputs x under technology $P \cup Q$.

From these two axioms, we derive several results. The first is a characterization of a class of rules which can be viewed as "generalized" numeraire rules. These rules, which we call *path-based measures*, work as follows. Each such measure is associated with a fixed and monotonic continuous path emanating from the origin and ending at some fixed point. For any pair of inputs and outputs, we scale the path so that the end of the path coincides with the vector of inputs, and then find the point of intersection of the path with the input set associated with that level of output. The further along this path, the more efficient the bundle of inputs. See Figure 5 for two examples of path-based measures.

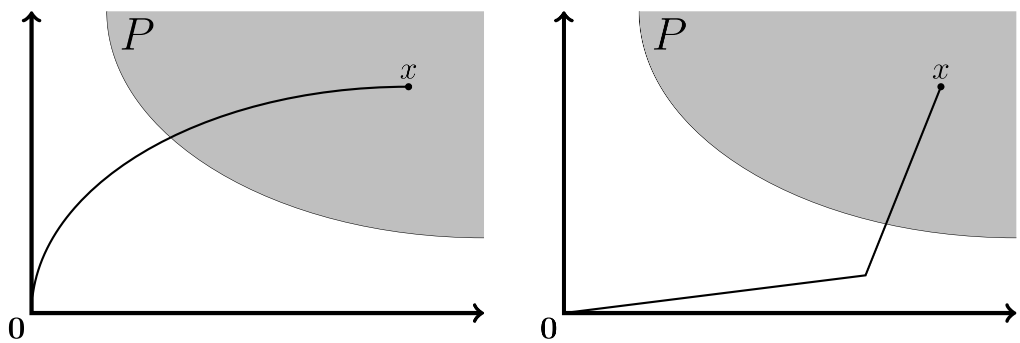


Figure 5: Path-Based Measures are determined by increasing paths

The characterization of path-based measures relies on three additional axioms: scale invariance, strong monotonicity, and *monotone continuity*, a basic continuity condition related to axioms found in the decision theory literature. (See Arrow (1971)).

The *scale invariance* axiom has a natural interpretation: it simply states that whether we measure inputs in pounds or kilograms makes no difference to the measurement of efficiency. To illustrate the *strong monotonicity* axiom, suppose that we have two technologies P and P'. We can say that it is unambiguously more efficient to produce an output under P' than under P if for any input x which can produce the output under P, it is possible to produce the same output under P' using strictly fewer of all resources contained in x. If technology becomes unambiguously more efficient yet we retain the previous level of inputs and outputs, then the axioms requires that there should be a strict decrease in the measured efficiency of the firm.

With cardinal versions of our axioms, Christensen, Hougaard, and Keiding (1999) characterize the coefficient of resource utilization along with a symmetry axiom. To this end, we describe such a symmetry axiom in our environment and establish an ordinal variant of their result. Our result differs from Christensen, Hougaard, and Keiding (1999) in a few technical respects. First, our characterization removes two of their axioms (continuity on rays and dominance). Second, we do not assume existence of a functional representation. Third, our theorem also applies on the domain of convex problems, while the Christensen, Hougaard, and Keiding (1999) result requires the existence of non-convex technologies. Lastly, their finite union property and conditional multiplicity axioms are somewhat weaker than choice consistency and planning consistency in that the former only apply in the case of indifference.

1.1 Related Literature

1.1.1 Previous axiomatic work on efficiency measurement

Previous axiomatic work on efficiency measurement generally takes a given technology as primitive, notable exceptions being Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999). An efficiency measure operates with respect to that prespecified technology. (See, for example, Färe and Lovell (1978) and Russell (1985).) In contrast, our approach specifies an efficiency measure which can work *across* technologies. The setup of our framework postulates an implicit independence axiom (only input sets matter). This amounts to an assumption that our measure is really a measure of *input* efficiency. A dual approach might study measures of output efficiency. To some degree, we require such a framework as our interpretation of technology may be different from preceding works. Our definition conceives of technology of the specific resources available to a given firm at a given point in time; it is the classical notion of a production possibility set. Other such definitions seek to understand whether society is operating at an efficient level, given the current state of the art (in a general equilibrium context, this would be the Minkowski sum of all individual production sets, as in Debreu (1959)).

Probably the closest work to ours is the axiomatic contribution of Christensen, Hougaard, and Keiding (1999). These authors introduce a cardinal framework which otherwise is very similar to ours. Our purpose has been to understand the "correct" departures from the coefficient of resource utilization in asymmetric environments, so a natural building block is their work which characterizes the coefficient. We offer a counterpart of their theorem in our ordinal framework (our Theorem 2) in order to highlight the connection. We also show how such a characterization can be established on the domain of convex sets.

1.1.2 Path-based measures

Aside from the aforementioned contributions of Debreu (1951) and Farrell (1957), the idea of using a path to compare alternatives relative to some set is not new, and seems to date back at least as far as Dupuit (1844). The classical reason to studying these objects was in order to cardinally measure changes in welfare. When comparing two consumption bundles, one can find the indifference set on which the second bundle lies, and then take a path-based measure based on the original consumption. The welfare change in such a measure is determined by the distance one would need to travel on this path. The paths considered in this literature were typically straight lines following an axis—effectively measuring utility using a numeraire.

Wold (1943a,b, 1944) illustrates a classical construction of utility functions (taught in most current economics textbooks) based on following a path from the origin and finding the point in which this path intersects a specific indifference curve. Allais (1952, 1981) suggests path-based rules as a method defining welfare change (analogous to compensating or equivalent variation). Luenberger (1992, 1996) also discusses generalized path-based rules as welfare measures.

Social choice and Nash bargaining theory are rife with path-based style rules. In particular, Kalai (1977) and Thomson and Myerson (1980) axiomatize Nash bargaining solution based on monotone paths.

Nevertheless, as far as we can tell, our characterization of path-based measures is entirely new, and is the first to rely on purely ordinal comparisons.

1.1.3 Mathematics and lattice homomorphisms

Formally, our two axioms, planning and choice consistency, imply that rules (for a fixed vector of inputs) are *lattice homomorphisms*, from a certain lattice of subsets (ordered by set inclusion) to the lattice of real numbers (with the typical ordering). Kreps (1979) seems to be the first to state an axiom analogous to choice consistency, albeit in an entirely different framework. He observed already that this axiom was necessary and sufficient (in a finite world) for a binary relation over sets to be generated by maximization of another binary relation over points. An analogue of this result plays an implicit role in the proof of our own result, and is the driving force behind the characterization of Hougaard and Keiding (1998), who derive necessary and sufficient conditions for an efficiency measure to be characterized by the minimization of a function (normalized by inputs) on the input set.

Miller (2008), Dimitrov, Marchant, and Mishra (2011), Chambers and Miller (2011), Leclerc and Monjardet (2011), Leclerc (2011), study variations of the planning and choice consistency axioms in other economic environments.

2 The model and results

A set $X \in \mathbb{R}^\ell$ is *comprehensive* if, for all $x, y \in \mathbb{R}^\ell$, $x \in X$ and $y \geq x$ implies that $y \in X$.[2][3] Let Σ denote the set of comprehensive and closed sets $P \subseteq \mathbb{R}^\ell_+$. A set $P \in \Sigma$ is referred to an *input set.* Let $\mathcal{P} \subseteq \Sigma \times \mathbb{R}^\ell_{++}$ such that $(P, x) \in \mathcal{P}$ only if $x \in P$. An ordered pair $(P, x) \in \mathcal{P}$ is referred to as and *efficiency measurement problem.* In particular, we are interested in studying two domains of efficiency measurement problems. First, let $\mathcal{P}' \subseteq \Sigma \times \mathbb{R}^\ell_{++}$ denote the *general domain*, for which $(P, x) \in \mathcal{P}'$ if and only if $x \in P$. Second, let $\mathcal{P}'' \subseteq \mathcal{P}'$ denote the *domain of convex problems*, for which $(P, x) \in \mathcal{P}''$ if and only if $(P, x) \in \mathcal{P}'$ and P is convex.

The two domains we consider are each historically of interest to economists. For example, convex technologies are particularly of interest in a general equi-

[2] Vector inequalities: $x \geq y$ if $x_i \geq y_i$ for all i, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_i > y_i$ for all i.

[3] Comprehensivity refers to free disposability. Free disposability may be unduly strong in the case of possible congestion effects. A generalization of this concept is that of "ray monotonicity," see (Färe, Grosskopf, and Lovell, 1987). Our measures continue to satisfy all of the postulated axioms when input sets are only required to be ray monotonic.

librium setting. The proof structure of our characterization results depends on the domain of interest.

An *ordinal efficiency measure* is a binary relation $\succeq$ on $\mathcal{P}$. We discuss several properties of ordinal efficiency measures.

The first axiom is standard: it merely states that the ranking should be complete and transitive. This axiom rules out the Malmquist Index and other non-transitive measures. (See Färe, Grosskopf, Norris, and Roos (1997).)

Weak order: The binary relation $\succeq$ is complete and transitive.

The second axiom was described in the introduction. It relates to a firm that must commit to producing without knowing which relevant technology will be feasible tomorrow.

Planning consistency: For all P, Q and all $x \in P \cap Q$, if $(P, x) \succeq (Q, x)$, then $(P \cap Q, x) \sim (P, x)$.

The third axiom, also described in the introduction, relates to a firm that has the option of choosing one, and only one, technology from which to produce. Note that this axiom is stated in a nonbinary fashion (that is, it refers to arbitrary finite collections P_i). This is so because on the domain of convex problems, it is not necessarily the case that $\bigcup_i P_i$ is a feasible input set.

Choice consistency: For all x and all finite collections P_i for which $x \in P_i$, if $\bigcup_i P_i \in \Sigma$ and $(P_i, x) \succeq (P_j, x)$ for all i, then $(\bigcup_i P_i, x) \sim (P_j, x)$.

Choice consistency is equivalent to the following on the general domain of problems. The proof is a simple induction argument.

Weak choice consistency: For all P, Q and all $x \in P \cap Q$, if $(P, x) \succeq (Q, x)$, then $(P \cup Q, x) \sim (Q, x)$.

The next property, strong monotonicity, states that as technology becomes unambiguously better, then remaining at current production levels must be considered worse. Note that a weak version of monotonicity is already implied by either planning or choice consistency.

Strong monotonicity: If $P \subseteq \text{int}Q$, then $(P,x) \succ (Q,x)$.[4]

For the next axiom, we need some basic definitions. For every $\lambda \in \mathbb{R}^{\ell}_{++}$ and $x \in \mathbb{R}^{\ell}$, define $\lambda * x \equiv (\lambda_1 x_1, ..., \lambda_\ell x_\ell)$. Similarly, $\lambda * P = \{\lambda * x : x \in P\}$.

Scale invariance states that the measure should be invariant to units of measurement of all commodities.

Scale invariance: For all $(P,x) \in \mathcal{P}$ and all $\lambda \in \mathbb{R}^{\ell}_{++}$, $(P,x) \sim (\lambda * P, \lambda * x)$.

Lastly, we introduce a basic continuity axiom.

Monotone Continuity: Let $\{F_i\}_{i\in\mathbb{N}} \subseteq \Sigma$ be a decreasing sequence of sets for which $\bigcap_{i\in\mathbb{N}} F_i \in \Sigma$. Let $E \in \Sigma$ and $x \in \bigcap_{i\in\mathbb{N}} F_i \cap E$. If $(E,x) \succeq (F_i,x)$ for all $i \in \mathbb{N}$, then $(E,x) \succeq (\bigcap_{i\in\mathbb{N}} F_i, x)$.

An *increasing path* is defined as a continuous mapping $g : [0,1] \to \mathbb{R}^{\ell}$ for which $g(0) = \mathbf{0}$, $g(1) = \mathbf{1}$ and for which $x > y$ implies $g(x) > g(y)$. A scale-invariant path-based measure is one for which there exists an increasing path g such that, for all $(P,x) \in \mathcal{P}$, $f(P,x) = \inf\{\beta : g(\beta) * x \in P\}$. An ordinal efficiency measure is *path-based* if it there exists a scale-invariant path-based measure f such that $(P,x) \succeq (Q,y)$ if and only if $f(P,x) \geq f(Q,y)$.

[4] In this paper, when we refer to interior we mean the relative interior. For $(P,x),(Q,x) \in \mathcal{P}$, we write that $P \subseteq \text{int}Q$ if there does not exist a continuous mapping $g : [0,1] \to R^{\ell}_{+}$ for which $g(\mathbf{0}) = 0$, $g(x) = 1$, and for which $g([0,1]) \cap (Q \setminus P) = \varnothing$.

Theorem 1. *On either the general domain or the domain of convex problems, a ordinal efficiency measure satisfies the weak order, planning consistency, choice consistency, monotone continuity, strong monotonicity and scale invariance axioms if and only if it is path-based. Furthermore, the six axioms are independent.*

We illustrate the set of path-based measures by means of examples which we believe to be new to this literature. Our first such example is the *lexicographic commodity ranking*. Suppose that the commodities are prioritized in terms of "importance," so that commodity 1 is more important than commodity 2, and so forth. Consider the path $g : [0,1] \to \mathbb{R}^\ell$ given by $g_i(x) = 0$ if $x \leq \frac{i-1}{\ell}$, $g_i(x) = \ell \cdot \left(x - \frac{i-1}{\ell}\right)$ if $\frac{i-1}{\ell} < x \leq \frac{i}{\ell}$, and $g_i(x) = 1$ if $\frac{i}{\ell} < x$.

The ordinal efficiency measure associated with this path compares two problems by the proportion of commodity ℓ that could be reduced without hurting production. If, for both problems, commodity ℓ could be eliminated without hurting production, the measure proceeds by comparing the proportion of commodity $\ell - 1$ that could be reduced without hurting production, and so forth.

Each ordering of the commodities implies a different rule. If we take the expectation with respect to all such paths (according to a uniform measure), the resulting path corresponds to the coefficient of resource utilization. By changing the weighting of the lexicographic orderings, we can generate a rule which incorporates different tradeoffs in the prioritization of different commodities. An example of lexicographic commodity rankings is shown in Figure 6. In this example, the lexicographic commodity ranking which prioritizes commodity one is depicted in red while the ranking which prioritizes commodity two is depicted in blue. Also shown are paths which result from convex combinations of the lexicographic commodity rankings.

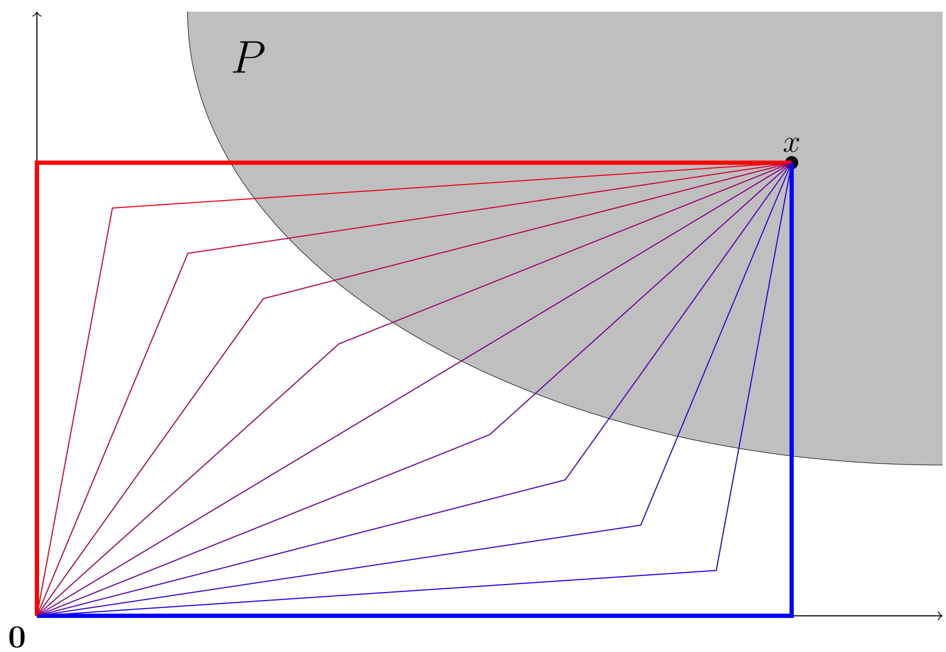


Figure 6: Lexicographic Paths and Convex Combinations

2.1 The Coefficient of Resource Utilization

The coefficient of resource utilization Debreu (1951) is a path-based measure associated with the straight line from the current inputs to the origin. To characterize this axiom we introduce a strong axiom, symmetry, which states that all commodities should be treated equally according to the measure. It forbids us from giving precedence to one commodity over another in terms of efficiency measurement.

For every permutation $\sigma : \{1, ..., \ell\} \to \{1, ..., \ell\}$, define $\sigma \circ x \equiv \left(x_{\sigma(1)}, ..., x_{\sigma(\ell)}\right)$. Similarly, $\sigma \circ P = \{\sigma \circ x : x \in P\}$.

Symmetry: For all $(P,x) \in \mathcal{P}$ and all permutations σ, $(P,x) \sim (\sigma \circ P, \sigma \circ x)$.

The *coefficient of resource utilization* of Debreu (1951) is the function $f_c : \mathcal{P} \to [0,1]$ given by $f_c(P,x) \equiv \inf\{\alpha : \alpha x \in P\}$.

Theorem 2. *There is a unique ordinal efficiency measure satisfying the axioms weak order, planning consistency, choice consistency, strong monotonicity, scale invariance, and symmetry on either the general domain or the domain of convex problems. It is represented by the coefficient of resource utilization; that is*

$$(P,x) \succeq (Q,y) \text{ if and only if } f_c(P,x) \geq f_c(Q,y). \tag{1}$$

Furthermore, the six axioms are independent.

3 Other domains

Certain environments may allow the negative production of certain commodities, for example, when firms may borrow against future production. In such an environment, it is reasonable to suppose that the set of possible input sets $\hat{\Sigma}$ consist of all comprehensive, nonempty, and closed sets $P \subseteq \mathbb{R}^\ell$ which are bounded below in the sense that there is a point $x \in \mathbb{R}^\ell$ for which $x \leq y$ for all $y \in P$. In this case, we can define our domain $\hat{\mathcal{P}}$ to include all problems (P,x) for which $P \in \hat{\Sigma}$ and $x \in P$.

Most of the axioms we previously described can be immediately described in this environment. We need to modify strong monotoncity, so that the interior operator is interpreted as the usual interior (not the relative interior). Scale invariance needs to be removed altogether. Instead, we suggest replacing scale invariance with the following axiom, which we label *translation invariance.* For $y \in \mathbb{R}^\ell$, we define $P + y \equiv \{x \in \mathbb{R}^\ell : x - y \in P\}$.

Translation invariance: For all $P, Q \in \hat{\Sigma}$, all $x \in P \cap Q$, and all $y \in \mathbb{R}^\ell$,

$$(P, x) \succeq (Q, x) \text{ if and only if } (P + y, x + y) \succeq (Q + y, x + y).$$

It is now easy to describe a counterpart of the family described in Theorem 1. We define an *unbounded increasing path* to be a continuous function $g : (-\infty, 0] \to \mathbb{R}^\ell$ which is strictly increasing, and satisfying $g(0) = 0$, and finally, for all $\alpha \in \mathbb{R}_+$, there exists $x \in (-\infty, 0]$ for which $\|g(x)\| = \alpha$. The last requirement simply states that the path is "unbounded below."

We can now define an analogue of the path-based measures. A translation-invariant path-based measure $f : \hat{\mathcal{P}} \to (-\infty, 0]$ is one for which there exists an unbounded increasing path g such that $f(P, x) = \inf\{\beta \leq 0 : g(\beta) + x \in P\}$. For example, the measures described by Chambers, Chung, and Färe (1996) are translation-invariant path-based measures whereby $g(\beta) = b\beta$, for some fixed $b \in \mathbb{R}^\ell_+$.

An ordinal efficiency measure on $\hat{\mathcal{P}}$ is *translation invariant path-based* if it there exists a translation-invariant path-based measure f such that $(P, x) \succeq (Q, y)$ if and only if $f(P, x) \geq f(Q, y)$.[5]

The following is the analogue of Theorem 1 where scale-invariance is replaced by translation-invariance. We state it without proof, as the details are almost identical to the proof of Theorem 1.

Theorem 3. *On $\hat{\mathcal{P}}$, an ordinal efficiency measure satisfies the weak order, planning consistency, choice consistency, monotone continuity, strong monotonicity and translation invariance axioms if and only if it is translation invariant path-based. Furthermore, the six axioms are independent.*

[5] An ordinal efficiency measure on $\hat{\mathcal{P}}$ is simply an ordinal ranking on $\hat{\mathcal{P}}$, just as it was for the domain $\mathcal{P}$.

4 Conclusion

This paper has introduced the notion of an ordinal efficiency measure. We have suggested that the ultimate interest of efficiency measurement is to compare alternative production plans. The comparative structure of such a problem suggests an ordinal approach, rather than a cardinal one. By so doing, we have been able to generate a large class of rules for measuring efficiency which depart from classical rules. The utility of these rules lies in their freedom to adjust efficiency measurement in order to accommodate tradeoffs between a lexicographic approach, and an approach where all commodities are treated equally (the coefficient of resource utilization).

Appendix

Proof of Theorem 1

To prove this theorem we make use of the following proposition.

Proposition 1. *Suppose that* $\succeq$ *satisfies planning consistency and monotone continuity. Then for every* $(P, x) \in \mathcal{P}$*, there exists some* $Q \subseteq \mathbb{R}^{\ell}_{+}$ *such that* $(P, x) \succeq (P', x)$ *if and only if* $Q \subseteq P'$*.*

To prove Proposition 1, we need the following lemma.

Lemma 1. *Suppose that* $\{E_\lambda\}_{\lambda \in \Lambda}$ *is an indexed family of closed subsets of* $\mathbb{R}^n$*. Then there exists a countable collection* $\{E_i\}_{i \in \mathbb{N}}$ *for which* $\bigcap_{i \in \mathbb{N}} E_i = \bigcap_{\lambda \in \Lambda} E_\lambda$*.*

Proof. Note that the closed subsets of $\mathbb{R}^n$, endowed with the Fell topology, is a compact metrizable space (Aliprantis and Border, 2007, 3.95). Let $\mathcal{K}$ denote the collection of finite subsets of Λ. Consider the collection $\mathcal{E} = \{\bigcap_{k \in K} E_k\}_{K \in \mathcal{K}}$. We claim that $\bigcap_{\lambda \in \Lambda} E_\lambda$ is a point in the closure of $\mathcal{E}$.

To see this, let U be a basic open neighborhood containing $\bigcap_{\lambda\in\Lambda} E_\lambda$. In particular, there is a $\mathbb{R}^n$ compact set L and $\mathbb{R}^n$ open sets $\{V_j\}_{j=1}^J$ for which $\bigcap_{\lambda\in\Lambda} E_\lambda \subseteq L^c$ and $\bigcap_{\lambda\in\Lambda} E_\lambda \cap V_j \neq \varnothing$. Clearly, every element $E \in \mathcal{E}$ satisfies $E \cap V_j \neq \varnothing$ (as $E \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda$). To see that there exists $E \in \mathcal{E}$ for which $E \subseteq L^c$, note that $L \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$, so that $\{E_\lambda^c\}_{\lambda\in\Lambda}$ is an open cover of L. Consequently, there is $K \in \mathcal{K}$ for which $L \subseteq \bigcup_{k\in K} E_k^c$, or $\bigcap_{k\in K} E_k \subseteq L^c$. So $\bigcap_{k\in K} E_k \in U$.

Now, let d be a metric generating the Fell topology, and let $K_1, K_2, ...,$ be elements of $\mathcal{K}$ for which $d\left(\bigcap_{k\in K_i} E_k, \bigcap_{\lambda\in\Lambda} E_\lambda\right) \leq 1/i$. By Aliprantis and Border (2007, 3.95), it follows that the topological lim inf of the sequence $\{\bigcap_{k\in K_i} E_k\}_{i\in\mathbb{N}}$ is $\bigcap_{\lambda\in\Lambda} E_\lambda$. This implies that if $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$, there exists $i \in \mathbb{N}$ for which $x \notin \bigcap_{k\in K_i} E_k$. This implies that $\bigcap_{i\in\mathbb{N}} \bigcap_{k\in K_i} E_k = \bigcap_{\lambda\in\Lambda} E_\lambda$. Now for each i, we simply enumerate the elements of K_i and concatenate the lists of elements, to construct a sequence $\{E_i\}_{i\in\mathbb{N}} = E_1, ..., E_{|K_1|}, E_{|K_1|+1}, ..., E_{|K_1|+|K_2|}, ...$ for which $\bigcap_{i\in\mathbb{N}} E_i = \bigcap_{\lambda\in\Lambda} E_\lambda$. □

We now proceed to prove Proposition 1.

Proof. Let $\succeq$ satisfy planning consistency and monotone continuity and let $(P, x) \in \mathcal{P}$. Define now $Q = \bigcap\{R : (P, x) \succeq (R, x)\}$. Clearly if $(P, x) \succeq (P', x)$, then $Q \subseteq P'$. Now let $(P', x) \in \mathcal{P}$ for which $Q \subseteq P'$. By Lemma 1, there exists a sequence $\{Q_i\}_{i\in\mathbb{N}} \subseteq \Sigma$ for which, for all $i \in \mathbb{N}$, (i) $x \in Q_i$ and (ii) $(P, x) \succeq (Q_i, x)$ satisfying $\bigcap_{i\in\mathbb{N}} Q_i = Q$. By planning consistency, we may without loss of generality choose this sequence to be decreasing with respect to set inclusion. By monotonicity, which is implied by planning consistency, $(P, x) \succeq (Q_i, x) \succeq (Q_i \cup P', x)$ for each $i \in \mathbb{N}$. By monotone continuity, it follows that $(P, x) \succeq (\bigcap_{i=1}^{\infty}(Q_i \cup P'), x)$. Because $\bigcap_{i=1}^{\infty}(Q_i \cup P') = Q \cup P' = P'$ it follows that $(P, x) \succeq (P', x)$. □

It is straightforward to verify that path-based measures satisfy the six axioms. Here we prove the converse. Let $\succeq$ satisfy the six axioms. We show that it must be path-based; that is, there exists a scale-invariant path-based measure f such that for every $(P,x),(Q,y) \in \mathcal{P}$, $(P,x) \succeq (Q,y)$ if and only if $f(P,x) \geq f(Q,y)$.

For $x \in \mathbb{R}^{\ell}_{++}$, let x^{-1} be the inverse of x, so that $x^{-1} * x \equiv \mathbf{1}$. By the scale invariance axiom, for all $(P,x) \in \mathcal{P}$, $(P,x) \sim (x^{-1} * P, \mathbf{1})$. Let $P' \equiv x^{-1} * P$ and let $Q' \equiv y^{-1} * Q$. Then, by transitivity, $(P,x) \succeq (Q,y)$ if and only if $(P', \mathbf{1}) \succeq (Q', \mathbf{1})$. This is entirely analogous to the proof of Theorem 2.

For any scale-invariant path-based measure there exists an increasing path g such that, for all $(P,x) \in \mathcal{P}$, $f(P,x) = \inf\{\beta : g(\beta) * x \in P\}$. It is obvious that $g(\beta)*x \in P$ if and only if $g(\beta) \in x^{-1}*P$. Therefore, for any scale-invariant path-based measure, $f(P,x) \geq f(Q,y)$ if and only if $f(P', \mathbf{1}) \geq f(Q', \mathbf{1})$. Thus, we must show that for every $(P', \mathbf{1}), (Q', \mathbf{1}) \in \mathcal{P}$, $(P', \mathbf{1}) \succeq (Q', \mathbf{1})$ if and only if $f(P', \mathbf{1}) \geq f(Q', \mathbf{1})$ for some scale-invariant path-based measure f.

Let $(P, \mathbf{1}) \in \mathcal{P}$. By Proposition 1 there exists $Q \subseteq \mathbb{R}^{\ell}_{+}$ such that $(P, \mathbf{1}) \succeq (P', \mathbf{1})$ if and only if $Q \subseteq P'$. For a set $S \subseteq \mathbb{R}^{\ell}$, define the **comprehensive hull** of S as $\mathcal{C}(S) \equiv \bigcup_{x \in S}\{y \in \mathbb{R}^{\ell} : y \leq x\}$, the smallest comprehensive set generated by S.

We claim that $x^* \equiv \bigwedge \{x : x \in Q\} \in Q$, and consequently that $Q = \mathcal{C}(\{x^*\})$. Assume, contrariwise, that this is false. (See Figure 7(a).) Because Q is closed and $x^* \notin Q$ there exists $x' \gg x^*$ such that $x' \notin Q$. (See Figure 7(b).) Because $x' \gg x^*$, there exists a set of ℓ points $x^1, ..., x^{\ell} \in Q$ such that, for all $i \leq \ell$, $x'_i \gg x^i_i \gg x^*$. (See Figure 7(c).) Define $H_i = \{x \in \mathbb{R}^{\ell}_{+} : x_i \geq x'_i\}$. (See Figures 7(d) and 7(e).) Let $i \leq \ell$. Because (a) $\mathbf{1} \in P \cap H_i$, (b) $P \cap H_i \in \Sigma$, and (c) $P \cap H_i \in \Sigma$ is convex whenever P is convex, it follows that $(P \cap H_i, \mathbf{1}) \in \mathcal{P}$. Next, because $x^i \notin H_i$, it follows that $Q \not\subseteq P \cap H_i$, and therefore that $(P, \mathbf{1}) \not\succeq (P \cap H_i, \mathbf{1})$. From the weak order axiom it follows that $(P \cap H_i, \mathbf{1}) \succ (P, \mathbf{1})$. From the choice consistency axiom it follows that $\left(\bigcup_{i=1}^{\ell} P \cap H_i, \mathbf{1}\right) \succ (P, \mathbf{1})$. Because $(P, \mathbf{1}) \succeq (P, \mathbf{1})$, it follows that $Q \subseteq P$. Furthermore, it is easily verified that $Q \subseteq \bigcup_{i=1}^{\ell} H_i$. (See Figure 7(f).) Consequently, $Q \subseteq \bigcup_{i=1}^{\ell} P \cap H_i$ and therefore $(P, \mathbf{1}) \succeq \left(\bigcup_{i=1}^{\ell} P \cap H_i, \mathbf{1}\right)$. This contradicts the weak order axiom, proving the claim.

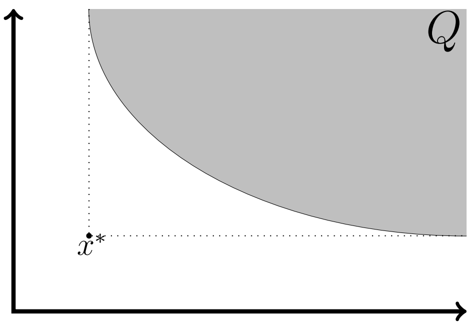
Thus for all $(P, \mathbf{1}) \in \mathcal{P}$ there exists a point $L(P) \in \mathbb{R}_+^{\ell}$ such that $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ for all $(Q, \mathbf{1}) \in \mathcal{P}$ such that $L(P) \in Q$. Define $G \equiv \bigcup_{(P,\mathbf{1}) \in \mathcal{P}} L(P)$. Note that $G \subseteq [0,1]^{\ell}$. First, for all $x, y \in G$, either $x \geq y$ or $y \geq x$. Otherwise, $(\mathcal{C}(\{x\}), \mathbf{1}) \in \mathcal{P}$ and $(\mathcal{C}(\{y\}), \mathbf{1}) \in \mathcal{P}$ would be unordered, violating weak order.

Next, note that for $(P, \mathbf{1}), (Q, \mathbf{1}) \in \mathcal{P}$, $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ if and only if $\min\{x \in G : x \in P\} \geq \min\{x \in G : x \in Q\}$.[6] In particular, $L(P) = \min\{x \in G : x \in P\}$. To see why, suppose by means of contradiction that there is $y \in G \cap P$ such that $y < L(P)$. By definition, $y = L(Q)$ for some Q. We know that $(Q, \mathbf{1}) \succeq (P, \mathbf{1})$ as $L(Q) \in P$ and $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$, as $L(P) \in Q$ (by comprehensivity of Q). Hence $(Q, \mathbf{1}) \sim (P, \mathbf{1})$. But since $L(P) \in \mathcal{C}(\{(L(P))\})$, we know that $(P, \mathbf{1}) \succeq (\mathcal{C}(\{L(P)\}), \mathbf{1})$, hence, $(Q, \mathbf{1}) \succeq (\mathcal{C}(\{L(P)\}), \mathbf{1})$, implying that $L(Q) \in \mathcal{C}(\{L(P)\})$, a contradiction.

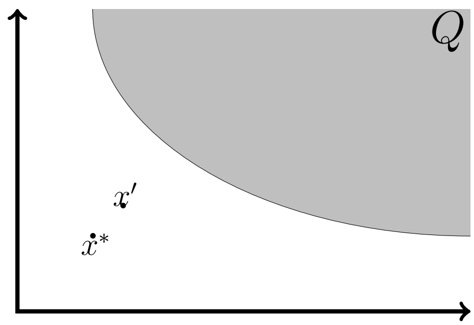
To prove the theorem we must show that there exists a continuous mapping $g : [0,1] \to \mathbb{R}^{\ell}$ for which $g(1) = \mathbf{1}$, for which $x > y$ implies $g(x) > g(y)$, and for which $g([0,1]) = G$.

For $\beta \in [0,1]$ let $H(\beta) \equiv \{x \in \mathbb{R}_+^{\ell} : \sum_{i=1}^{\ell} x_i = \ell\beta\}$. Define $g(\beta) \equiv G \cap H(\beta)$. Note that for all $\beta \in [0,1]$, the points in $H(\beta)$ are unordered with respect to $\leq$; hence $g(\beta)$ is at most single-valued. Because $G \subseteq [0,1]^{\ell} \subseteq$

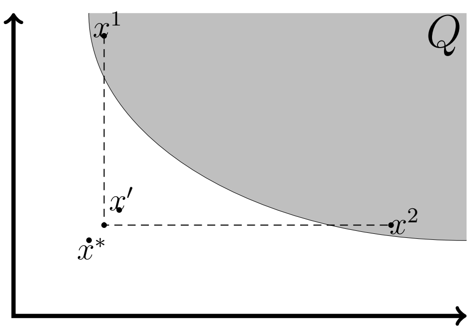
[6] The minimum refers to a minimum with respect to $\leq$.



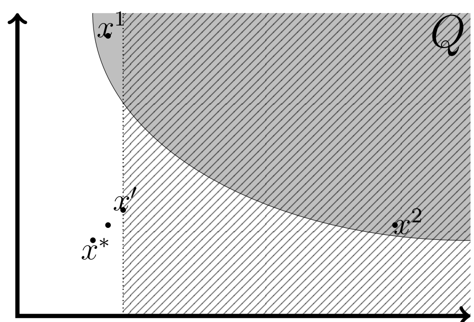
(a) Because Q is not the comprehensive hull of a single point, $x^* \equiv \bigwedge\{x \in Q\} \notin Q$.



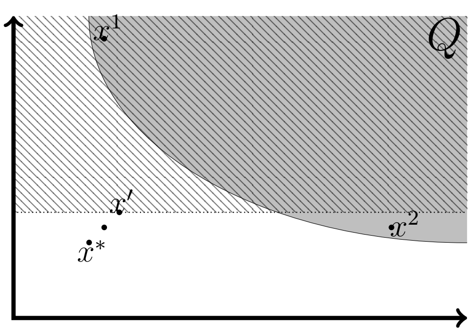
(b) Because $x^* \notin Q$, there is a point $x' \gg x^*$ such that $x' \notin Q$



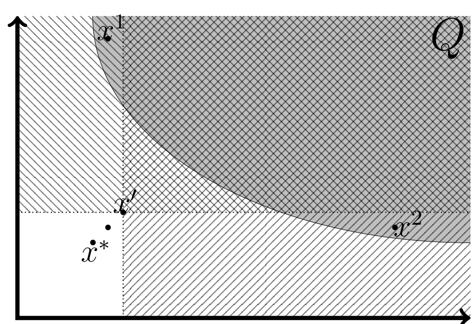
(c) Thus we can find points x^1 and x^2 such that $x' \gg x^1 \wedge x^2 \gg x^*$



(d) $H_1 \equiv \{x : x_1 \geq x'_1\}$. Note that $x^1 \notin H_1$



(e) $H_2 \equiv \{x : x_2 \geq x'_2\}$. Note that $x^2 \notin H_2$



(f) $Q \subseteq H_1 \cup H_2$

Figure 7: Steps in Proof of Theorem 1

$\cup_\beta H(\beta)$ it follows that $g([0,1]) \subseteq G$. For all $x \in G$, let $h(x) \equiv \frac{1}{\ell}\sum_{i=1}^{\ell} x_i$. Then clearly $x = L(H(h(x)))$, so therefore $G \subseteq g([0,1])$. Consequently, $g([0,1]) = G$. Furthermore, $g(0) = \mathbf{0}$ and $g(1) = \mathbf{1}$.

It remains to be shown that g is continuous. First we show that G is connected. Suppose contrariwise that G is not connected. Then there is a $\beta \in [0,1]$ for which $g(\beta) = \varnothing$. Let $K(\beta) \equiv \mathcal{C}(H(\beta))$, and note that $\alpha > \beta$ implies that $K(\alpha) \subseteq \text{int}K(\beta)$. Because $(K(\beta), \mathbf{1}) \in \mathcal{P}$ for all $\beta \in [0,1]$, it follows that $L(K(\beta)) \in \text{int}\mathcal{C}(H(\beta))$. Consequently, $K(h(L(K(\beta)))) \subseteq \text{int}K(\beta)$. By strong monotonicity, it follows that $(K(h(L(K(\beta)))), \mathbf{1}) \succ (K(\beta), \mathbf{1})$. By definition of L, since $L(K(\beta)) \in K(h(L(K(\beta)))$, we know that $(K(\beta), \mathbf{1}) \succeq (K(h(L(K(\beta)))), \mathbf{1})$, a contradiction.

To see that the path is continuous, let $\beta_k \to \beta^*$ be a sequence. We want to show that $g(\beta_k) \to g(\beta^*)$. Let U be a neighborhood of $g(\beta^*)$. Because U is a neighborhood of $g(\beta^*)$, there exists a sufficiently small $\varepsilon > 0$ such that if $|x_i - g_i(\beta^*)| < \varepsilon$ for all i, then $x \in U$. Now consider the interval $(\beta^* - \varepsilon, \beta^* + \varepsilon)$. By the monotonicity of g, for any $\beta \in (\beta^* - \varepsilon, \beta^* + \varepsilon)$, we know that $|g_i(\beta) - g_i(\beta^*)| < \varepsilon$. Since there are all but a finite number of β^k in U, the result is proved.

That the axioms are independent is proven below.

Proof of Theorem 2

It is clear that the coefficient of resource utilization represents an ordinal efficiency measure that satisfies these properties. We show that any ordinal efficiency measure that satisfies these properties must be represented by the coefficient of resource utilization. This is sufficient to prove uniqueness. Let $\succeq$ be an ordinal efficiency measure satisfying the six axioms, and let $(P, x), (Q, y) \in \mathcal{P}$

be efficiency measurement problems. We show that statement (1) must be true.

For $x \in \mathbb{R}^\ell_{++}$, let x^{-1} be the inverse of x, so that $x^{-1} * x = \mathbf{1}$. By the scale invariance axiom, $(P, x) \sim (x^{-1} * P, \mathbf{1})$ and $(Q, y) \sim (y^{-1} * Q, \mathbf{1})$. Thus by transitivity, $(P, x) \succeq (Q, y)$ if and only if $(x^{-1} * P, \mathbf{1}) \succeq (y^{-1} * Q, \mathbf{1})$. Because the coefficient of resource utilization is scale-invariant, $f_c(P, x) = f_c(x^{-1} * P, \mathbf{1})$ and $f_c(Q, y) = f_c(y^{-1} * Q, \mathbf{1})$. Thus statement (1) is true if $(x^{-1} * P, \mathbf{1}) \succeq (y^{-1} * Q, \mathbf{1})$ if and only if $f_c(x^{-1} * P, \mathbf{1}) \geq f_c(y^{-1} * Q, \mathbf{1})$ for any pair problems (P, x) and (Q, y). Without loss of generality, it is sufficient to show that $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ if and only if $f_c(P, \mathbf{1}) \geq f_c(Q, \mathbf{1})$ for any pair of problems $(P, \mathbf{1})$ and $(Q, \mathbf{1})$.

For any $\beta \in [0, 1]$, define $\mathcal{K}(\beta) \equiv \{x \in \mathbb{R}^\ell_+ : x_i \geq \beta \text{ for all } i\}$. For any input set $P \in \Sigma$, let $\beta(P) \equiv \inf\{\beta : \mathcal{K}(\beta) \subseteq P\}$. It is clear that $f_c(P, \mathbf{1}) = \beta(P)$. Thus to prove statement (1), it is sufficient to show that $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ if and only if $\beta(P) \geq \beta(Q)$ for any two problems $(P, \mathbf{1})$ and $(Q, \mathbf{1})$. Note that by strong monotonicity $\beta(P) \geq \beta(Q)$ if and only if $(\mathcal{K}(\beta(P)), \mathbf{1}) \succeq (\mathcal{K}(\beta(Q)), \mathbf{1})$. We show that for any problem $(P, \mathbf{1})$, $(P, \mathbf{1}) \sim (\mathcal{K}(\beta(P)), \mathbf{1})$. This is sufficient to prove statement (1), as it implies (by transitivity) that $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ if and only if $(\mathcal{K}(\beta(P)), \mathbf{1}) \succeq (\mathcal{K}(\beta(Q)), \mathbf{1})$. There are two cases.

The General Domain: For any $\beta \in [0, 1]$, define $\mathcal{K}_i(\beta) \equiv \{x \in \mathbb{R}^\ell_+ : x_i \geq \beta\}$. Then $\cap_{i=1}^{\ell} \mathcal{K}_i(\beta) = \mathcal{K}(\beta)$. For $j > 1$, let σ_{1j} be the permutation such that $\sigma_{1j}(1) = j$, $\sigma_{1j}(j) = 1$, and $\sigma_{1j}(k) = k$ for $k \neq 1, j$. Note that $\sigma_{1j} \circ \mathcal{K}_1(\beta) = \mathcal{K}_j(\beta)$. By symmetry, $(\mathcal{K}_1(\beta), \mathbf{1}) \sim (\sigma_{1j} \circ \mathcal{K}_1(\beta), \sigma_{1j} \circ \mathbf{1})$ and therefore $(\mathcal{K}_1(\beta), \mathbf{1}) \sim (\mathcal{K}_j(\beta), \mathbf{1})$ for all $j > 1$. By planning consistency and an induction argument it follows that $(\mathcal{K}_1(\beta), \mathbf{1}) \sim \left(\cap_{i=1}^{k} \mathcal{K}_i(\beta), \mathbf{1}\right)$ for every $k \leq \ell$ and therefore $(\mathcal{K}_1(\beta), \mathbf{1}) \sim (\mathcal{K}(\beta), \mathbf{1})$. Similarly, by choice consistency it follows that $\left(\bigcup_{i=1}^{\ell}\mathcal{K}_i(\beta),\mathbf{1}\right)\sim(\mathcal{K}_1(\beta),\mathbf{1})$, so that $\left(\bigcup_{i=1}^{\ell}\mathcal{K}_i(\beta),\mathbf{1}\right)\sim(\mathcal{K}(\beta),\mathbf{1})$. Now, let $(P,\mathbf{1})\in\mathcal{P}'$. For all $P\in\Sigma$, because P is closed and comprehensive it follows that $\mathcal{K}(\beta(P))\subseteq P$, and by construction of the sets $\mathcal{K}_i$ it follows that $P\subseteq\bigcup_{i=1}^{\ell}\mathcal{K}_i(\beta(P))$. Therefore, from monotonicity (implied by planning consistency) it follows that $(P,\mathbf{1})\sim(\mathcal{K}(\beta(P)),\mathbf{1})$.

The Domain of Convex Problems: For any $\beta\in[0,1]$, define $R(\beta)\equiv\{x:\sum_{i=1}^{\ell}x_i\geq\ell\beta\}$, and define $R_i(\beta)\equiv R(\beta)\cap\mathcal{K}_i(\beta)$. Note that (a) $\sigma_{1j}\circ R_1(\beta)=R_j(\beta)$ for all $j>1$, (b) $\cap_{i=1}^{\ell}R_i(\beta)=\mathcal{K}(\beta)$, and (c) $\cup_{i=1}^{\ell}R_i(\beta)=R(\beta)$. By symmetry, $(R_1(\beta),\mathbf{1})\sim(\sigma_{1j}\circ R_1(\beta),\sigma_{1j}\circ\mathbf{1})$ and therefore $(R_1(\beta),\mathbf{1})\sim(R_j(\beta),\mathbf{1})$ for all $j>1$. By planning consistency and an induction argument it follows that $(R_1(\beta),\mathbf{1})\sim\left(\cap_{i=1}^{k}R_i(\beta),\mathbf{1}\right)$ for every $k\leq\ell$ and therefore $(R_1(\beta),\mathbf{1})\sim(\mathcal{K}(\beta),\mathbf{1})$. Similarly, by choice consistency it follows that $\left(\bigcup_{i=1}^{\ell}R_i(\beta),\mathbf{1}\right)\sim(R_1(\beta),\mathbf{1})$, so that $(R(\beta),\mathbf{1})\sim(\mathcal{K}(\beta),\mathbf{1})$. Now, let $(P,\mathbf{1})\in\mathcal{P}''$. By symmetry, $(P,\mathbf{1})\sim\left(\bigcap_{\sigma}(\sigma\circ P),\mathbf{1}\right)$. For all $P\in\Sigma$, $\mathcal{K}(\beta(P))\subseteq\bigcap_{\sigma}(\sigma\circ P)\subseteq R(\beta(P))$, so it follows from monotonicity (implied by planning consistency) that $(P,\mathbf{1})\sim\left(\bigcap_{\sigma}(\sigma\circ P),\mathbf{1}\right)\sim(\mathcal{K}(\beta(P)),\mathbf{1})$.

That the axioms are independent is proven below.

Independence of the Axioms

We present six ordinal efficiency measures. It is simple to verify that the first five violate each violate one of weak order, planning consistency, choice consistency, strong monotonicity and scale-invariance while satisfying the other six axioms, and that the sixth measure violates both symmetry and monotone continuity while satisfying the remaining five axioms. This is sufficient to prove that both sets of axioms are independent.

Weak order: Define a measure as follows. For $(P,x),(Q,x)\in\mathcal{P}$, let

$(P,x) \succeq (Q,x)$ if and only if $f_c(P,x) \geq f_c(Q,x)$. For $(P,x),(Q,y) \in \mathcal{P}$ such that $x \neq y$, let $(P,x) \sim (Q,y)$ if and only if (i) $(Q,y) = (\lambda * Q, \lambda * y)$ for some $\lambda \in \mathbb{R}^\ell_{++}$ or (ii) $(Q,y) = (\sigma \circ Q, \sigma \circ y)$ for some permutation σ. This measure violates weak order but satisfies planning consistency, choice consistency, strong monotonicity, scale-invariance, symmetry, and monotone continuity.

Planning consistency: For $(P,x) \in \mathcal{P}$, let $f_2(P,x) \equiv 1 - \max\left\{\beta \leq 1 : \left\{y \in \mathbb{R}^\ell_+ : \frac{\|y-x\|}{\|x\|} \leq \beta\right\} \subseteq P\right\}$, and consider the ordinal efficiency measure $\succeq$ for which $(P,x) \succeq (Q,y)$ if and only if $f_2(P,x) \geq f_2(Q,y)$. This measure violates planning consistency but satisfies weak order, choice consistency, strong monotonicity, scale invariance, symmetry, and monotone continuity.

Choice consistency: For $(P,x) \in \mathcal{P}$, let $f_3(P,x) \equiv \max\left\{\beta \leq 1 : \left\{y \in \mathbb{R}^\ell_+ : \frac{\|y\|}{\|x\|} \leq \beta\right\} \not\subseteq P\right\}$, and consider the ordinal efficiency measure $\succeq$ for which $(P,x) \succeq (Q,y)$ if and only if $f_3(P,x) \geq f_3(Q,y)$. This measure violates choice consistency but satisfies weak order, planning consistency, strong monotonicity, scale invariance, symmetry, and monotone continuity.

Strong monotonicity: For $(P,x) \in \mathcal{P}$, let $f_4(P,x) \equiv 1$, and consider the ordinal efficiency measure $\succeq$ for which $(P,x) \succeq (Q,y)$ if and only if $f_4(P,x) \geq f_4(Q,y)$. This measure clearly violates strong monotonicity but trivially satisfies weak order, planning consistency, choice consistency, scale invariance, symmetry, and monotone continuity.

Scale invariance: For $(P,x) \in \mathcal{P}$, let $f_5(P,x) \equiv \inf\{\alpha : \alpha\mathbf{1} \in P\}$, and consider the ordinal efficiency measure $\succeq$ for which $(P,x) \succeq (Q,y)$ if and only if $f_5(P,x) \geq f_5(Q,y)$. This measure is not scale invariant but satisfies weak order, planning consistency, choice consistency, strong monotonicity, symmetry, and monotone continuity.

Monotone continuity and Symmetry: Let $g : [0,1] \to \mathbb{R}^\ell$ be defined by $g_i(x) = 0$ if $x \leq \frac{i-1}{\ell}$, $g_i(x) = \ell \cdot \left(x - \frac{i-1}{\ell}\right)$ if $\frac{i-1}{\ell} < x \leq \frac{i}{\ell}$, and $g_i(x) = 1$ if $\frac{i}{\ell} < x$. For $(P,x) \in \mathcal{P}$, let $f_6(P,x) = \sup\{\alpha : x * g(\alpha) \notin \text{int}P\}$. This measure violates monotone continuity and symmetry but satisfies weak order, planning consistency, choice consistency, and scale invariance.[7]

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[7] Any ordinal efficiency measure which satisfies either of the monotone continuity and symmetry axioms in addition to the other five must be a path-based measure.

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