Supplementary Notes to Unscheduled Appointments

Full Proofs

There are a few basic equalities that will be used throughout the proofs. First, let $\beta = (\beta_0, \beta_1, ..., \beta_n)$ be an arbitrary $n+1$ dimensional vector. Then:

$$\sum_{i=0}^{n} \binom{n}{i} (1-F(x))^i F(x)^{n-i} \beta_i = \sum_{i=0}^{m} \sum_{j=0}^{n-m} \binom{m}{i} \binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j} \tag{2}$$

Proof. By Vandermonde's identity, $\binom{n}{i} = \sum_{j=0}^{i} \binom{m}{i-j} \binom{n-m}{j}$.

Thus the left hand side of (2) is equal to $\sum_{i=0}^{n} \sum_{j=0}^{i} \binom{m}{i-j} \binom{n-m}{j} (1-F(x))^i F(x)^{n-i} \beta_i$.

If we replace i with $i+j$, then this becomes $\sum_{i+j=0}^{n} \sum_{j=0}^{i+j} \binom{m}{i} \binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j}$.

We can rewrite this expression as $\sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{m}{i} \binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j}$.

Because $\binom{a}{b} = 0$ for $a < b$, this is equivalent to $\sum_{i=0}^{m} \sum_{j=0}^{n-m} \binom{m}{i} \binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j}$. □

$$\kappa(x) = \lambda(x,x) + m(1-F(x)) \tag{3}$$

Proof. Recall that $\kappa(x) = \sum_{i=0}^{n} \binom{n}{i} (1-F(x))^i F(x)^{n-i} \min\{m, i\}$.

By expression (2), this equals $\sum_{i=0}^{m} \sum_{j=0}^{n-m} \binom{m}{i} \binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \min\{m, i+j\}$.

Rearranging terms, thus becomes:

$$\sum_{i=0}^{m} \binom{m}{i} (1-F(x))^i F(x)^{m-i} \sum_{j=0}^{n-m} \binom{n-m}{j} (1-F(x))^j F(x)^{n-m-j} (\min\{m-i, j\} + i).$$

This last expression is equivalent to $\lambda(x,x) + m(1-F(x))$. □

$$\frac{d}{dx} E[v|v \geq x] = \frac{f(x)}{1-F(x)} (E[v|v \geq x] - x) \tag{4}$$

Proof. Recall that $E[v|v \geq x] = \int_x^\infty \frac{vf(v)}{1-F(x)} dv$, or $\frac{1}{1-F(x)} \int_x^\infty vf(v)dv$. Using the chain rule,

$$\frac{d}{dx} E[v|v \geq x] = \frac{f(x)}{(1-F(x))^2} \int_x^\infty vf(v)dv - \frac{1}{1-F(x)} xf(x), \text{ or } \frac{f(x)}{1-F(x)} (E[v|v \geq x] - x).$$

□

The proofs of the following thre statements are straightforward and left to readers.

$$\kappa'(x) = \sum_{i=0}^{n} \binom{n}{i} (1-F(x))^i F(x)^{n-i} \left(\frac{n-i}{F(x)} - \frac{i}{1-F(x)} \right) f(x) \min\{m, i\} \tag{5}$$

$$\left.\frac{dv^*}{dc}\right|_{c=0} = \frac{n(1-F(p))}{\kappa(p)} \tag{6}$$

$$\left.\frac{d\hat{v}}{dc}\right|_{c=0} = \frac{(n-m)(1-F(p))}{\lambda(p,p)} \tag{7}$$

Proof of Theorems 2.1 and 2.2.

If we take the derivative of $W_A(c)$ with respect to the transportation cost, c, we get:

$$W_A'(c) = \kappa'(p+c)\left(E[v|v \geq p+c] - c\right) + \kappa(p+c)\left(\left[\tfrac{d}{d(p+c)} E[v|v \geq p+c]\right] - 1\right).$$

Evaluated at $c = 0$, this becomes:

$$W_A'(0) = \kappa'(p)E[v|v \geq p] + \kappa(p)\left[\tfrac{d}{dp}E[v|v \geq p]\right] - \kappa(p).$$

If we take the derivative of $W_U(c)$ with respect to the transportation cost, c, we get:

$W_U'(c) = \kappa'(v^*)\frac{dv^*}{dc}E[v|v \geq v^*] + \kappa(v^*)\left[\frac{d}{dv^*}E[v|v \geq v^*]\right]\frac{dv^*}{dc} - n(1-F(v^*)) + nf(v^*)c\frac{dv^*}{dc}$.

Evaluated at $c = 0$, this becomes:

$$W_U'(0) = \kappa'(p)\left.\tfrac{dv^*}{dc}\right|_{c=0} E[v|v \geq p] + \kappa(p)\left[\tfrac{d}{dp}E[v|v \geq p]\right]\left.\tfrac{dv^*}{dc}\right|_{c=0} - n(1-F(v^*)).$$

Using expression (6) and simplifying, we have:

$$W_U'(0) = \left(\kappa'(p)E[v|v \geq p] + \kappa(p)\left[\tfrac{d}{dp}E[v|v \geq p]\right] - \kappa(p)\right)\left.\tfrac{dv^*}{dc}\right|_{c=0}.$$

Or, $W_U'(0) = W_A'(0)\left.\frac{dv^*}{dc}\right|_{c=0}$. Expression (6) is greater than one, thus

$W_A'(0) \geq W_U'(0)$ if and only if $W_A'(0) \leq 0$.

This is equivalent to: $\kappa(p)\frac{f(p)}{1-F(p)}\left(E[v|v \geq p] - p\right) \leq \kappa(p) - \kappa'(p)E[v|v \geq p]$.

Evaluated at $p = 0$ this is: $\kappa(0)f(0)E[v|v \geq 0] \leq \kappa(0)$.

This is true if and only if: $f(0)\int_0^\infty xf(x)dx \leq 1$, which proves Theorem 2.1.

By assumption, $\Gamma'(p) \leq 0$, which implies that $\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) \leq 1$.

This implies that $\kappa(p)\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) \leq \kappa(p)$.

It follows from the fact that $\kappa'(p) \leq 0$ and $E\left[v|v \geq p\right] > 0$ that $W_A'(0) \geq W_U'(0)$ for all prices p.

Furthermore, because $\kappa'(p) < 0$ for all $p > 0$, it follows that $W_A'(0) > W_U'(0)$ for all prices $p > 0$.

At $p = 0$, the fact that $\kappa'(0) = 0$ implies that $W_A'(0) > W_U'(0)$ if and only if

$\kappa(p)\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) < \kappa(p)$, which is true if and only if $\Gamma'(0) < 0$.

This proves Theorem 2.2.

Proof of Lemma 2.3.

If $c = 0$, $\hat{v} = p$, and thus $W_S(0) = m\left(1 - F(p)\right)E[v|v \geq p] + \lambda(p,p)E[v|v \geq p]$.

By expression (3), this equals $\kappa(p)E[v|v \geq p] = W_U(0)$.

Proof of Theorem 2.4.

If we take the derivative of $W_S(c)$ with respect to the transportation cost, c, we get:

$$W_S'(c) = -mf(p+c)\left(E\left[v|v \geq p+c\right] - c\right)$$

$$+m\left(1 - F(p+c)\right)\left(\frac{f(p+c)}{1-F(p+c)}\left(E\left[v|v \geq p+c\right] - p - c\right) - 1\right) + \frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}E\left[v|v \geq \hat{v}\right]$$

$$+\lambda\left\{\hat{v}, p+c\right\}\frac{f(\hat{v})}{1-F(\hat{v})}\left(E\left[v|v \geq \hat{v}\right] - \hat{v}\right)\frac{d\hat{v}}{dc} - (n-m)\left(1 - F(\hat{v})\right) + (n-m)f(\hat{v})\frac{d\hat{v}}{dc}c.$$

After simplifying:

$$W_S'(c) = -mpf(p+c) - m\left(1 - F(p+c)\right) - (n-m)\left(1 - F(\hat{v})\right) + (n-m)f(\hat{v})\frac{d\hat{v}}{dc}c$$

$$-\lambda(\hat{v}, p+c)\frac{\hat{v}f(\hat{v})}{1-F(\hat{v})}\frac{d\hat{v}}{dc} + E\left[v|v \geq \hat{v}\right]\left[\frac{\lambda\{\hat{v},p+c\}f(\hat{v})}{1-F(\hat{v})}\frac{d\hat{v}}{dc} + \frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\right].$$

At $c = 0$, if we substitute expression (7), this becomes:

$$W_S'(0) = E\left[v|v \geq p\right]\left[(n-m)f(p) + \frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0}\right] - npf(p) - n\left(1 - F(p)\right).$$

From the proof of Theorem 2.1 and substituting expression (6), we get:

$$W_U'(0) = n\left(f(p) + \frac{(1-F(p))\kappa'(p)}{\kappa(p)}\right)E\left[v|v \geq p\right] - npf(p) - n\left(1 - F(p)\right).$$

Thus, $W_U'(0) \geq W_S'(0)$ if and only if $\frac{n(1-F(p))\kappa'(p)}{\kappa(p)} \geq \frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0} - mf(p)$.

Note that $\frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\} = \sum_{i=0}^{m}\binom{m}{i}\left(1 - F(p+c)\right)^{i} F(p+c)^{m-i}\left(\frac{m-i}{F(p+c)} - \frac{i}{1-F(p+c)}\right) f(p+ c)$

$\sum_{j=0}^{n-m}\binom{n-m}{j}\left(1 - F(\hat{v})\right)^{j} F(\hat{v})^{n-m-j}\min\{m-i, j\} + \sum_{i=0}^{m}\binom{m}{i}\left(1 - F(p+c)\right)^{i} F(p+ c)^{m-i}$

$\sum_{j=0}^{n-m}\binom{n-m}{j}\left(1 - F(\hat{v})\right)^{j} F(\hat{v})^{n-m-j}\left(\frac{n-m-j}{F(\hat{v}} - \frac{j}{1-F(\hat{v})}\right)\frac{d\hat{v}}{dc} f(\hat{v})\min\{m-i, j\}$.

Evaluated at $c = 0$, and using expression (7), this becomes:

$$\frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0} = f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1 - F(p)\right)^{i+j} F(p)^{n-i-j}$$
$$\left(\left(\frac{m-i}{F(p)} - \frac{i}{1-F(p)}\right) + \left(\frac{n-m-j}{F(p)} - \frac{j}{1-F(p)}\right)\frac{(n-m)(1-F(p))}{\lambda(p,p)}\right)\min\{m-i, j\}$$

This simplifies to:

$$\frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0} = f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1 - F(p)\right)^{i+j} F(p)^{n-i-j}$$
$$\left(\left(\frac{m-i}{F(p)} - \frac{i}{1-F(p)}\right) + \left(\frac{n-m-j}{F(p)} - \frac{j}{1-F(p)}\right)\frac{(n-m)(1-F(p))}{\lambda(p,p)}\right)\min\{m-i, j\}$$

Thus $W_U'(0) \geq W_S'(0)$ if and only if:

$$\frac{n(1-F(p))\kappa'(p)}{\kappa(p)} \geq f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1 - F(p)\right)^{i+j} F(p)^{n-i-j}$$
$$\left(\left(\frac{m-i}{F(p)} - \frac{i}{1-F(p)}\right) + \left(\frac{n-m-j}{F(p)} - \frac{j}{1-F(p)}\right)\frac{(n-m)(1-F(p))}{\lambda(p,p)}\right)\min\{m-i, j\} - mf(p)$$

Multiplying each side by $\lambda(p,p)\kappa(p)$:

$\lambda(p,p)n\left(1 - F(p)\right)\kappa'(p) \geq f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1 - F(p)\right)^{i+j} F(p)^{n-i-j}\min\{m- i, j\}$

$$\left(\left(\frac{m-i}{F(p)}-\frac{i}{1-F(p)}\right)\lambda(p,p)\kappa(p)+\left(\frac{n-m-j}{F(p)}-\frac{j+\frac{m}{n-m}}{1-F(p)}\right)\kappa(p)(n-m)\left(1-F(p)\right)\right)$$

Combining statements (5) and (2):

$$\kappa'(p)=\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)f(p)\min\left\{m,i+j\right\}.$$

Rearranging terms and applying statement (2):

$$\kappa'(p)=f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)\min\left\{m-i,j\right\}+$$

$$mf(p)\left(1-F(p)\right)\sum_{i=0}^{n-1}\binom{n-1}{i}\left(1-F(p)\right)^{i}F(p)^{n-1-i}\left(\frac{n-1-i}{F(p)}-\frac{i+1}{1-F(p)}\right).$$

Note that $\sum_{i=0}^{n-1}\binom{n-1}{i}\left(1-F(p)\right)^{i}F(p)^{n-1-i}\left(\frac{n-1-i}{F(p)}-\frac{i+1}{1-F(p)}\right)=\frac{-1}{1-F(p)}$. Thus:

$\kappa'(p)=f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)\min\left\{m-i,j\right\}-$
$mf(p)$.

Substituting for $\kappa'(p)$ and dividing each side by $f(p)$, it follows that $W_U'(p)\geq W_S'(p)$ if and only if:

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$$

$$\left(\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)\lambda(p,p)-m\right)n\left(1-F(p)\right)\geq$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$$

$$\left(\left(\frac{m-i}{F(p)}-\frac{i}{1-F(p)}\right)\lambda(p,p)\kappa(p)+\left(\frac{n-m-j}{F(p)}-\frac{j+\frac{m}{n-m}}{1-F(p)}\right)\kappa(p)(n-m)\left(1-F(p)\right)\right)$$

Multiplying each side by $F(p)$:

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$$

$$\left(n^2\left(1-F(p)\right)\lambda(p,p)-n(i+j)\lambda(p,p)-mn\left(1-F(p)\right)F(p)\right)\geq$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$$

$$\left(m\lambda(p,p)-\frac{i\lambda(p,p)}{1-F(p)}+(n-m)^2\left(1-F(p)\right)-(n-m)j-mF(p)\right)\kappa(p)$$

Rearranging terms:

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$$

$(n^2(1-F(p))\lambda(p,p) - mn(1-F(p))F(p) + (mF(p) - m\lambda(p,p) - (n-m)^2(1-F(p)))\kappa(p)) \geq$

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$

$\left(n(i+j)\lambda(p,p) - \frac{i\lambda(p,p)\kappa(p)}{1-F(p)} - (n-m)j\kappa(p)\right)$

Using the substitution in statement (3) and cancelling terms:

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$

$\left[-m(n(1-F(p))-\kappa(p))^2 - mF(p)(n(1-F(p))-\kappa(p))\right] \geq$

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$

$\left(n(i+j)(\kappa(p)-m(1-F(p))) - \frac{i(\kappa(p)-m(1-F(p)))\kappa(p)}{1-F(p)} - (n-m)j\kappa(p)\right)$

Note that $\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}\min\{m-i,j\}X_{ij}$

$= \sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}(m-i)X_{ij}$

$- \sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}(m-i-j)X_{ij}.$

It follows that $W_U'(p) \geq W_S'(p)$ if and only if:

$-m\left[(n(1-F(p))-\kappa(p))^2 + F(p)(n(1-F(p))-\kappa(p))\right]$

$\left[mF(p) - \sum_{i=0}^{m}\binom{n}{i}(1-F(p))^{i}F(p)^{n-i}(m-i)\right] \geq$

$m\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}$

$\left((i+j)n(\kappa(p)-m(1-F(p))) - i\left(\frac{\kappa(p)^2}{1-F(p)} - m\kappa(p)\right) - j(n-m)\kappa(p)\right)$

$-\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}i$

$\left((i+j)n(\kappa(p)-m(1-F(p))) - i\left(\frac{\kappa(p)^2}{1-F(p)} - m\kappa(p)\right) - j(n-m)\kappa(p)\right)$

$-\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}(m-i-j)(i+j)n(\kappa(p)-m(1-F(p))) +$

$\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}(m-i-j)i\left(\frac{\kappa(p)^2}{1-F(p)} - m\kappa(p)\right) +$

$\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}(1-F(p))^{i+j}F(p)^{n-i-j}(m-i-j)j(n-m)\kappa(p)$

Or:

$$-m\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i)\right]\geq$$

$$-m^2(n(1-F(p))-\kappa(p))^2$$

$$-m(1-F(p))\sum_{i=1}^{m}\sum_{j=0}^{n-m}\binom{m-1}{i-1}\binom{n-m}{j}(1-F(p))^{i-1+j}F(p)^{n-i-j}$$

$$\left((i+j)n(\kappa(p)-m(1-F(p)))-i\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)-j(n-m)\kappa(p)\right)-$$

$$(1-F(p))\sum_{k=1}^{m}\binom{n-1}{k-1}(1-F(p))^{k-1}F(p)^{n-k}(m-k)n^2(\kappa(p)-m(1-F(p)))+$$

$$(1-F(p))\sum_{i=1}^{m}\sum_{j=0}^{m-i}\binom{m-1}{i-1}\binom{n-m}{j}(1-F(p))^{i-1+j}F(p)^{n-i-j}(m-i-j)m\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)+$$

$$(1-F(p))\sum_{i=0}^{m}\sum_{j=1}^{m-i}\binom{m}{i}\binom{n-m-1}{j-1}(1-F(p))^{i+j-1}F(p)^{n-i-j}(m-i-j)(n-m)^2\kappa(p)$$

Or:

$$-m\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i)\right]\geq$$

$$-m^2(n(1-F(p))-\kappa(p))^2$$

$$-m(1-F(p))\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}$$

$$\left((k+j+1)n(\kappa(p)-m(1-F(p)))-(k+1)\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)-j(n-m)\kappa(p)\right)$$

$$-(1-F(p))\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)n^2(\kappa(p)-m(1-F(p)))$$

$$+(1-F(p))\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)m\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+(1-F(p))\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)(n-m)^2\kappa(p)$$

Or:

$$-m\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i)\right]\geq$$

$$-m^2(n(1-F(p))-\kappa(p))^2$$

$$-m(1-F(p))\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}(k+j)n(\kappa(p)-m(1-F(p)))$$

$$-m(1-F(p))\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}n(\kappa(p)-m(1-F(p)))$$

$$+m(1-F(p))\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}k\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+m(1-F(p))\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+m(1-F(p))\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}j(n-m)\kappa(p)$$

$$+m(n(1-F(p))-\kappa(p))^2\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)$$

Dividing each side by m:

$$-\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i)\right]\geq$$

$$-m(n(1-F(p))-\kappa(p))^2$$

$$-(n-1)(1-F(p))^2\sum_{k=1}^{n-1}\binom{n-2}{k-1}(1-F(p))^{k-1}F(p)^{n-1-k}n(\kappa(p)-m(1-F(p)))$$

$$-(1-F(p))n(\kappa(p)-m(1-F(p)))$$

$$+(m-1)(1-F(p))^2\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-2}{k-1}\binom{n-m}{j}(1-F(p))^{k+j}F(p)^{n-1-k-j}\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+(1-F(p))\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+(n-m)(1-F(p))^2\sum_{k=0}^{m-1}\sum_{j=1}^{n-m-1}\binom{m-1}{k}\binom{n-m-1}{j-1}(1-F(p))^{k+j}F(p)^{n-1-k-j}(n-m)\kappa(p)$$

$$+(n(1-F(p))-\kappa(p))^2\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)$$

Or:

$$-\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^i F(p)^{n-i}(m-i)\right]\geq$$

$$-m\left(n\left(1-F(p)\right)-\kappa(p)\right)^2$$

$$-(n^2-n)\left(1-F(p)\right)^2\kappa(p)+m(n^2-n)\left(1-F(p)\right)^3$$

$$-\left(1-F(p)\right)n\kappa(p)+nm\left(1-F(p)\right)^2$$

$$+(m-1)\left(1-F(p)\right)\kappa(p)^2-(m^2-m)\left(1-F(p)\right)^2\kappa(p)$$

$$+\kappa(p)^2-m\left(1-F(p)\right)\kappa(p)$$

$$+(n-m)^2\left(1-F(p)\right)^2\kappa(p)$$

$$+\left(n\left(1-F(p)\right)-\kappa(p)\right)^2\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)$$

Or:

$$-\left[mF(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+mF(p)^2\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$+\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^i F(p)^{n-i}(m-i)\geq$$

$$-m\left(n\left(1-F(p)\right)-\kappa(p)\right)^2F(p)$$

$$+\left(m\left(1-F(p)\right)-\kappa(p)\right)\left(n\left(1-F(p)\right)-\kappa(p)\right)F(p)$$

$$+\left(n\left(1-F(p)\right)-\kappa(p)\right)^2\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)$$

Or:

$$\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^i F(p)^{n-i}(m-i)\geq$$

$$\left(m-\kappa(p)\right)\left(n\left(1-F(p)\right)-\kappa(p)\right)F(p)$$

$$+\left(n\left(1-F(p)\right)-\kappa(p)\right)^2\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)$$

Note that $m-\kappa(p) = m-\lambda(p,p)-m\,(1-F(p)) = \sum_{i=0}^{m}\binom{n}{i}(1-F(p))^{i}F(p)^{n-i}(m-i)$. Thus:

$$(n\,(1-F(p))-\kappa(p))^2\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^{i}F(p)^{n-i}(m-i) \geq$$

$$(n\,(1-F(p))-\kappa(p))^2 + (n\,(1-F(p))-\kappa(p))^2\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)$$

It follows that $W_U'(p) \geq W_S'(p)$ if and only if:

$$\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^{i}F(p)^{n-i}(m-i)-\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i) \geq 0$$

Using the identity $\binom{n-1}{i} = \binom{n}{i}-\binom{n-1}{i-1}$, this equation becomes:

$$\sum_{i=0}^{m}\binom{n-1}{i}(1-F(p))^{i}F(p)^{n-i}(m-i)$$

$$+\sum_{i=0}^{m}\binom{n-1}{i-1}(1-F(p))^{i}F(p)^{n-i}(m-i)-\sum_{i=0}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i) \geq 0$$

This reduces to:

$$\sum_{i=0}^{m}\binom{n-1}{i}(1-F(p))^{i}F(p)^{n-i}(m-i)-\sum_{i=0}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i+1}(m-i) \geq 0$$

Because $\binom{n-1}{-1} = 0$, and substituting j for $i-1$, we get:

$$\sum_{i=0}^{m}\binom{n-1}{i}(1-F(p))^{i}F(p)^{n-i}(m-i)-\sum_{j=0}^{m-1}\binom{n-1}{j}(1-F(p))^{j}F(p)^{n-j}(m-j-1)$$

$$= \sum_{i=0}^{m}\binom{n-1}{i}(1-F(p))^{i}F(p)^{n-i} \geq 0.$$

This last statement is clearly true, and the inequality holds strictly if and only if $p > 0$.