CONTESTS WITH THREE OR MORE HETEROGENEOUS AGENTS

SÉRGIO O. PARREIRAS[♯] AND ANNA RUBINCHIK[♮]

Abstract. We study monotone equilibrium behavior in contests with observable effort (bid) where three or more participants have distinct risk attitudes and the monetary value for the prize of each is drawn independently from a distinct distribution. These differences can either cause a player to drop out, that is always choose zero effort regardless of his valuation, or use "all-or-nothing" strategies with discontinuous effort choice. Neither *complete drop-out* nor *discontinuous bidding* with finitely many gaps is consistent with pure strategy monotone Bayesian-Nash equilibrium in a contest with either ex-ante identical players or only two distinct participants.

1. Introduction

Contests are all around us. Students striving to be the best in their class, employees awaiting promotion, sportsmen fighting for a gold medal, R&D firms racing to capture monopoly profits, researchers competing for grants, politicians running in a presidential election — all can be viewed as players in games with just one take-it-all winner. Typically, the rest of the participants have to absorb the cost of the invested effort and get no prize.

We model a contest as a game of incomplete information with a common prior, where the beliefs of the players about the rivals' values for the prize reflect the commonly observable differences. Such flexibility allows us to capture many interesting scenarios. Indeed, rarely

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[♯]Dept. of Economics, UNC at Chapel Hill. sergiop@unc.edu.

[♮]Dept. of Economics, University of Haifa, Mount Carmel, Haifa, 31905; Israel. annarubinchik@gmail.com.

do the contestants look alike: their background, previous experience, gender, age and other observable characteristics vary, and so, each can be perceived as being different from his opponents. These differences, naturally, affect a player's choice of effort.

To the best of our knowledge, theoretical literature on all-pay auctions under incomplete information so far has focused either on just two contestants or on ex-ante identical competitors.[1] In the latter case, the effect of observable differences is impossible to analyze; while the former, we show, is rather special.

Under complete information and with identical cost of effort, as in Hillman and Riley (1989); Baye et al. (1993) only the two individuals with the highest values for the prize enter the competition, while the rest drop out (generically). Differences in the costs of effort, as in Siegel (2009), can induce higher participation. Heterogeneity under incomplete information has different implications.

Take three or more contestants who differ both by their risk attitudes and rival's beliefs about their valuations for the prize. An *underdog*, whose possible upper valuation is sufficiently low, might never play the upper bid chosen by his rivals. Moreover, he can even *completely drop out*, i.e., bid zero independently of his valuation, even if there is a non-negligible *ex-ante* probability that he is the highest valuation contestant. Moderately risk-averse avoid investing as much effort as their very-risk-averse rivals, while the latter might adopt almost an "all-or-nothing" discontinuous strategy. So, in this framework one can rationalize a wide range of behaviors that are inconsistent with an equilibrium where a contestant is facing only one rival or a group of ex-ante clones of oneself, as we show in sections 3 and 4.

To consider an illustration of the drop-out results, take a contest with 3 risk-neutral participants. In this case the value that a player associates with the prize is equivalent to his ability (or the inverse of the effort cost). Adrian is the underdog: his rivals, Brian and Cindy, believe his ability is a draw from a uniform distribution with support $[0, 4]$. Brian and Cindy are considered strong: their abilities are believed to be drawn from a uniform distribution on $[0, 100]$. It might seem obvious that, given Adrian would never bid more than 4, his rivals with ability of more than 50 *should* bid above 4. However, the argument supporting this conjecture is not immediate. Given that positive bids can not be chosen with a strictly positive probability in equilibrium (otherwise it would be possible to discontinuously increase the probability of winning

[1]See, e.g., Lizzeri and Persico (1998); Krishna and Morgan (1997); Fibich et al. (2006); Amann and Leininger (1996).

at an arbitrarily small price), bidding at the top *assures* anyone of the victory. Hence, if all three contestants believe that some $\overline{b} \leq 4$ is the top equilibrium bid common to all, why should anyone bid higher? Here is a rough calculation that illustrates the reason. Consider an equilibrium in which Brian and Cindy use the same non-decreasing strategy. When Brian's valuation is 50, he defeats Cindy with probability $\frac{1}{2}$, which is the upper bound on his winning probability, thus his expected payoff is at most 25. However, if he deviates by bidding $5 > \overline{b}$, he wins for sure and receives 45. It follows that the top bid $\overline{b}$ can not be (weakly) below 4, and hence can not be common to all in such an equilibrium.

Prop. 1 provides a threshold for a player such that whenever the support of his valuations falls below it, he will not choose the top bid played by at least two of his rivals. This threshold, of course, depends on his beliefs about the valuations of the rivals (more precisely, on the upper boundary of the respective supports) and the risk attitudes of all.[2] Prop. 2 shows how strong a player's rivals should be to assure that he will *never* place a positive bid, thus staying out of the competition altogether. If our Adrian is facing other rivals, however, with the abilities drawn from a uniform distribution with a closer support (say, $[0, 3]$ and $[0, 6]$), then he will not drop out completely, see fig. 2.

To illustrate the discontinuity result take 3 contestants whose values have common support, but let their risk attitudes vary. Adrian is infinitely risk averse, while Brian and Cindy have some moderate risk-attitudes. Adrian, unlike Brian and Cindy, would never place a strictly positive bid associated even with a minimal risk of losing as then his payoff is negative, so he can only bid at the bottom (zero) or at the top. Unfortunately, in this case, there is no equilibrium.[3] Prop. 3 is formulated for well-defined risk attitudes, when the equilibrium exists, but the underlying argument is similar to the extreme case: our Adrians

[2]Example 1 illustrates the calculation for the risk-neutral case.

[3]If the top bid is strictly below the top value, then, whenever Adrian's value falls in the interval between the highest bid and the top value, he has to bid at the top. But this can not be consistent with an equilibrium, as top bid can not be chosen with a strictly positive probability (as observed above). The only remaining scenario is when the top bid is equal to the the top value, but in this case the expected profit of the player with the highest valuation is zero, and therefore, profits of the rest should be no higher, hence, zero as well. This, too is inconsistent with an equilibrium: if the strategy maps an open interval of values to an interval of strictly positive bids, and all contestants in that interval get zero payoff, then, given strict monotonicity of the strategies (lemma 1) and the single-crossing property of a payoff (ft. 5), a contestant with a value in that interval who is prescribed to bid in the interior can get a strictly positive payoff by deviating to a lower bid. Clearly, in no equilibrium all contestants use only extreme bids either.

are sufficiently risk averse, so that for a range of low valuations their best choice is the safe bid, zero.

One of the benefits of our investigation is to provide a framework to analyze the effect of *group composition* on elicited effort, participation and aggressiveness of the players. In this context, group composition embeds both the distribution of risk preferences within a group as well as beliefs of the opponents about one's valuation for the prize (or ability).

Here is the plan. Section 2 describes the environment and the equilibrium bidding functions. The next two sections contain our main results: sufficient conditions for drop-out behavior are formulated in section 3, and those for discontinuous bidding are in section 4. Section 5 contains two possible reasons for an aggressive behavior in contests: differences in risk attitudes and dominance of distributions of valuations. We discuss the related literature and some implications of our findings in the conclusions, section 6. The proofs missing in the text are in the appendix.

2. The Model

There are $N \geq 3$ risk-averse individuals competing for a single prize. The prize is allocated to a contestant who demonstrates the top performance. In case of a tie the winner is selected randomly according to a non-degenerate distribution, p, describing the likelihood of getting the prize for all those at the top.[4] One's performance fully reflects the effort (bid), so it is fully observable, as is standard in the all-pay auction literature.

The payoff to winner k is $u_k(v_k - b)$, where $b \geq 0$ is the (monetized) effort he exerts, and v_k is his monetary equivalent for the prize. A loser $j \neq k$ bidding b gets $u_j(-b)$, as the (investment of) effort is irreversible. We assume that all $u_i : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable, concave, strictly increasing, with $u_i' > 0$. Hence the contestants can have different risk attitudes.

Each contestant knows his value from winning and shares the same beliefs about the values of the common rivals with any other contestant. The values, they presume, are distributed independently, but not necessarily identically, $V_i \sim F_i$ on $[\underline{v}, \overline{v}_i]$, $\underline{v} \geq 0$, with the density f_i, continuous and bounded away from zero for all $v \in [\underline{v}, \overline{v}_i]$.

[4]That is, $p(\mathbf{b})$ satisfies $p_i(\mathbf{b}) \geq 0$, $\sum p_i(\mathbf{b}) = 1$ and, $p_i(\mathbf{b}) > 0$ if and only if $b_i = \hat{b}$, where $\mathbf{b} = (b_1, \ldots, b_N)$ is the bid profile and, $\hat{b} = \max_i b_i$ is the highest observed bid.

In case all are risk-neutral, F_i also reflects the shared belief about i's abilities, and so even in this case the players can be distinct at the outset of the game; this is the second source of ex-ante heterogeneity.

Contestant i can formulate beliefs about his rivals' behavior, that is, the likelihood of $j \neq i$, to exert effort x_j, and the resulting distribution of the maximal effort of the rivals, $\overline{x} = \max_{j \neq i}\{x_j\}$. His probability of winning then is $W_i(b) = \Pr(b > \overline{x}) + \Pr(b = \overline{x})\, p_i(b, x_{-i})$, and his expected *interim* payoff should be $\Pi_i(b|v_i) = W_i(b)u_i(v_i - b) + (1 - W_i(b))\, u_i(-b)$.

2.1. **Equilibrium.** A strategy for individual i is a Lebesgue-measurable function that maps valuations into effort levels, $b_i : [\underline{v}, \overline{v}_i] \to \mathbb{R}_+$. We restrict attention to (pure-strategy) equilibria in which contestants with higher valuations for the prize expend weakly higher effort, or, simply, bid higher. Existence of a pure strategy, monotone Bayes-Nash equilibrium follows from Athey (2001, thm. 7, p. 881).

To describe such an equilibrium, it is convenient to express the winning probability in terms of players' strategies.

Definition 1. For any player $i = 1, \ldots, N$ let $\phi_i : b \mapsto v$ be a (generalized) *inverse bid function*: $\phi_i(b) \stackrel{\text{def}}{=} \max\left(\underline{v}, \sup\{v : b_i(v) \leq b\}\right)$.

Lemma 1. *Equilibrium bid function b_i of any contestant i is strictly increasing on $(\phi_i(0), \overline{v}_i]$.*

Proof. Assume to the contrary that b_i is not strictly increasing. Since b_i is non-decreasing, there must exist an interval of types where b_i assumes a constant value, say $\hat{b}$. This means that contestant i bids $\hat{b}$ with strictly positive probability and given the tie-breaking rule, the winning probability of any other contestant must jump discontinuously at $\hat{b}$. As a result, it is never optimal for any other contestant to place bids in the interval $(\hat{b} - \varepsilon, \hat{b}]$ for some $\varepsilon > 0$, which implies that contestant i should never bid $\hat{b}$. ■

Lemma 2. *The lowest bid in an equilibrium is zero.*

Proof. Assume to the contrary, that $b_i(\underline{v}) = \beta = \min_j b_j(\underline{v}) > 0$. Then no rival $j \neq i$ should bid β with positive probability so, i's winning probability is constant on $(0, \beta]$ and, he would be better off by reducing his bid. ■

First, observe that by the two lemmas ties happen with strictly positive probability only at zero in an equilibrium.

Second, generalized inverse bid functions, ϕ_i, are continuous for $b > 0$ as follows from lemma 1, and at $b = 0$ they are continuous by construction; in addition they are differentiable almost everywhere as they are bounded and non-decreasing.

Given the first observation, one can express the equilibrium probability of winning for all the contestants. Let $G_i(b) \overset{\text{def}}{=} \text{Prob}\,[b_i(V_i) \le b] = F_i(\phi_i(b))$ be the probability that contestant i bids at or below b. By independence, the probability of winning by contestant i who bids b can be expressed as the product of cumulative distributions of equilibrium bids of the rivals, $W_i\,(b) = \prod_{j \neq i} G_j(b)$.

Monotonicity of the bidding strategies implies that the winning probability W_i of player i is differentiable for almost every bid $b > 0$, allowing then to describe the best response of i using the first order conditions for maximization of the interim payoff, $\Pi_i(b|v_i)$, $v_i \in [\underline{v}, \overline{v}_i]$. Almost any choice of optimal bid $b > 0$, must satisfy[5]

$$
\begin{aligned}
(2.1)\qquad MB_i(b) &= MC_i\,(b)\,, \text{ where} \\
MB_i\,(b) &= [u_i(v_i - b) - u_i(-b)]\, W_i'\,(b) \text{ and} \\
MC_i\,(b) &= u_i'(-b)\,(1 - W_i\,(b)) + u_i'(v_i - b) W_i\,(b)\ .
\end{aligned}
$$

If the marginal benefit, MB_i, is below marginal cost, MC_i, for any choice of $b \in (0, v_i]$, then in an equilibrium the type v_i should *drop out*, $b_i(v_i) = 0$.

Now one can describe equilibrium behavior using the equilibrium bid density functions, and the set of active participants for each effort $b > 0$, that is, the set of contestants, who choose b for at least one of their types:[6]

$$J(b) \overset{\text{def}}{=} \left\{ i \in \{1, .., N\}\, | g_i(b) > 0 \text{ or } \lim_{x \to b} g_i(x) = +\infty \right\}\ .$$

For later use define also the size (cardinality) of $J(b)$, the corresponding set of i's rivals and its size:

Notation. $K(b) = \#J(b)$, $J_i(b) = J(b) \backslash i$, $K_i(b) = \#J_i(b)$.

In a model with ex-ante identical or only two distinct participants, for almost all b, $J(b)$ either includes all the participants, or is empty: all the players bid in the same range in equilibria there. However, in the presence of more than two ex-ante distinct contestants this is no longer true, as we demonstrate in the next two sections. Although we

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[5]Since contestants are weakly risk-averse, $MB_i\,(b) - MC_i\,(b)$ is strictly increasing in v_i for $b > 0$, so $\Pi_i(b|v_i)$ satisfies the strict single-crossing property.

[6]An equivalent formulation for the set of active bidders is $J(b) = \{i \,|\, b_i(v) = b \text{ for some } v \in [\underline{v}, \overline{v}_i] \text{ and } b_i \text{ is differentiable at } v\}$.

are unable to provide an explicit algorithm to determine $J(b)$ from the primitives of the model, we show how to use the first order conditions to check whether a contestant with value $v > \underline{v}$ should place bid $b > 0$ in a given equilibrium. For that we construct a *participation test*, [7] which uses a necessary condition for active bidding, that is, non-negativity of the implied rate of growth of the bidding probability, g_i/G_i, at b. This test, which is sufficient to rule out participation, but not to establish one, is used to derive our main results in the next two sections.

3. Participation, group composition and shared beliefs

The effect of heterogeneity on participation in full information contests was discussed in the introduction. Here we start with a benchmark result highlighting the importance of players' beliefs.

Lemma 3 (Common Top Bid)**.** *Any two contestants i and j with identical utilities and with common support of valuations, place the same bid at the top, $b_i(\overline{v}) = b_j(\overline{v})$.*

Proof. Assume to the contrary, there exists $b_j(\overline{v}) = \beta < b_i(\overline{v}) = \tilde{\beta}$, so $G_i(\beta) < G_j(\beta) = 1$. Given the identity, $W_i(b)G_i(b) = W_j(b)G_j(b)$, this implies $W_i(\beta) > W_j(\beta)$ and $W_j(\tilde{\beta}) = W_i(\tilde{\beta})$. Thus, given the utilities are identical, $\Pi_i(\beta|\overline{v}) > \Pi_j(\beta|\overline{v})$ and $\Pi_i(\tilde{\beta}|\overline{v}) = \Pi_j(\tilde{\beta}|\overline{v})$ which contradicts the revealed preferences, $\Pi_j(\beta|\overline{v}) \geq \Pi_j(\tilde{\beta}|\overline{v})$ and $\Pi_i(\tilde{\beta}|\overline{v}) \geq \Pi_i(\beta|\overline{v})$. ■

It follows that in a symmetric model with any number of participants the top bid is the same for all in *any* equilibrium.[8] This conclusion is also true for a contest with two different participants.[9] Indeed, why should any of them bid above the top bid of the only rival?

With at least three heterogeneous contestants some might drop out either partially (prop. 1) or completely (prop. 2).

The intuition for both results stems from the first order conditions, eq. (2.1). They imply that a player of type v bids b only if the marginal winning probability, $W_i'(b)$, is above the *required (or demanded) marginal probability of winning*,

$$D_i(b,v) \stackrel{\text{def}}{=} W_i(b)B_i(b,v) + (1 - W_i(b))L_i(b,v) \tag{3.1}$$

$$B_i(b,v) \stackrel{\text{def}}{=} \frac{u_i'(v-b)}{u_i(v-b) - u_i(-b)}, \quad L_i(b,v) \stackrel{\text{def}}{=} \frac{u_i'(-b)}{u_i(v-b) - u_i(-b)}$$

[7] See cor. 1 in appendix A.1.

[8] Hence this holds for the set up in Fibich et al. (2006).

[9] As in Amann and Leininger (1996).

Clearly, at the top, $D_i(\overline{b},\overline{v}) = B_i(\overline{b},\overline{v})$, so if $B_i(\overline{b},\overline{v}) > W_i'(\overline{b})$ player i finds it profitable to reduce his bid: the marginal bid cut is not going to change the winning probability by too much, thus the saved bid cost will outweight the foregone marginal benefit.[10]

Naturally, the required winning probability depends on player i's attitudes towards risk and the monetary value he attaches to winning.

Though Arrow-Pratt's coefficient of absolute risk-aversion, $r_i(x) = -u_i''(x)/u_i'(x)$, depends on the wealth level, *sufficient* conditions for the drop out will be formulated in terms of its extreme values, which are well-defined by our assumptions on utilities.

Notation. $\overline{r}_i = \max_{x\in[-\overline{v}_i,\overline{v}_i]} r_i(x)$ and $\underline{r}_i = \min_{x\in[-\overline{v}_i,\overline{v}_i]} r_i(x)$.

The advantage of this approach is that the two key elements of the required marginal winning probability for a player with a constant risk aversion r are independent of wealth, and therefore, the fully "sunk" bid. Indeed, using Pratt's (1964) representation of utilities,[11] $B_i(b,v) = \frac{\exp\left(-\int_{-b}^{v-b} r dz\right)}{\int_{-b}^{v-b}\exp\left(-\int_{-b}^{y} r dz\right)dy} = \frac{r}{e^{rv}-1}$ and, similarly, $L_i(b,v) = \frac{-r}{e^{-rv}-1}$. Hence, the two terms can be expressed using $\Phi(r,v) \stackrel{\text{def}}{=} \frac{r}{e^{rv}-1}$, which decreases in both real arguments.

Let us fix player i for the next two propositions.

The first one states that if the lowest required marginal winning probability of a player at the top, $\Phi(\overline{r}_i,\overline{v}_i)$, is higher than roughly the average of the (upper bounds of the) rest, then he should abstain from ever picking the top bid in any equilibrium where at least two of his rivals bid at the top.

Proposition 1 (Partial drop-out)**.** *Consider an equilibrium where $K_i(\overline{b}) \geq 2$, $\overline{b} = \max_j b_j(\overline{v}_j)$. If*

$$\frac{1}{K_i(\overline{b})-1}\sum_{j\in J_i(\overline{b})}\Phi(\underline{r}_j,\overline{v}_j) < \Phi(\overline{r}_i,\overline{v}_i)$$

then $b_i(\overline{v}_i) < \overline{b}$.

Note the assumptions of the claim are consistent with player i having the highest realized value among all active participants, yet in such an equilibrium he should not choose the sure-winner bid $\overline{b}$. It might

[10]Notice that $D_i(\overline{b},\overline{v}) < W_i'(\overline{b})$ is not a sufficient condition for $b_i(\overline{v}) = \overline{b}$. Unlike other auction formats (e.g., first-price or second-price) in the all-pay auction, local incentive constraints do not guarantee that the bid is individually rational, that is, it is possible to have $MB_i(b) \geq MC_i(b)$ and yet $\Pi_i(b|v) < 0$.

[11]That is, $u_i(x) = \int_0^x \exp\left(-\int_0^y r_i(z)dz\right)dy$.

happen that the top bid is above his upper valuation, but it is not the only possible scenario.

Clearly, heterogeneity is necessary for the hypothesis of the proposition: either the risk preferences of the contestants or their upper valuations have to be different. In the two examples below we examine the effect of the two sources of heterogeneity separately.

Example 1 (UNDERDOGS AVOID THE TOP)**.** If all the contestants are risk-neutral, but the distributions of their values have distinct respective supports, $[\underline{v}, \overline{v}_i]$ then the assumption of prop. 1 becomes:

$$\frac{1}{K_i(\overline{b})-1}\sum_{j\in J_i(b)}\frac{1}{\overline{v}_j}<\frac{1}{\overline{v}_i}$$

.

Example 2 (RISK-TOLERANT AVOID THE TOP)**.** If the preferences of the contestants exhibit constant relative risk aversion, CARA, so $r_i(x) = \rho_i$, and their valuations have common support, $[\underline{v}, \overline{v}]$, then the assumption of prop. 1 translates to:

$$\frac{1}{K_i(\overline{b})-1}\sum_{j\in J_i(\overline{b})}\frac{\rho_j}{e^{\rho_j\overline{v}}-1}<\frac{\rho_i}{e^{\rho_i\overline{v}}-1}.$$

We use the CARA set-up to construct a numerical example, see fig. 1. As one would expect, more risk-averse rivals are more aggressive in their bidding at the top since they want to insure themselves against losing. The figure shows that with such rivals the risk-neutral contestant finds it more profitable to shade his bid and bear the risk of losing rather than play at the top.

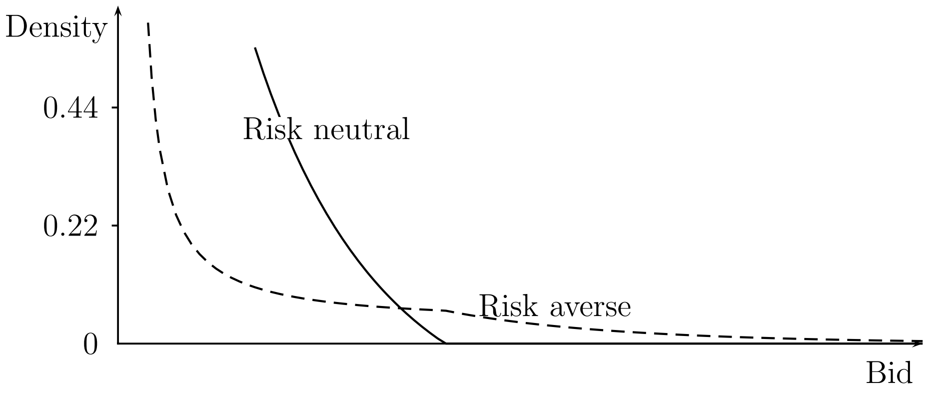

Fig. 1: Bid density of five identical contestants with $\rho_j = 1$ who face a risk neutral one. All have uniform valuations drawn from $[0, 1]$.

Remarkably, further increasing the gap in risk aversion is not going to generate a complete drop-out. Indeed, given all the distributions have the same support, the risk-neutral player could have bid at the top and get the prize for sure, thus enjoying a positive payoff, as other players do. However, even when his valuation is the highest, he chooses a different bid indicating his payoff must be even higher and thus, strictly positive. Hence so should be the bid, which means the player actively participates in the contest for at least some of his valuations.

Not surprisingly, to assure the complete drop-out, stronger conditions are needed.

Proposition 2 (Complete Drop-out). *Assume* $\underline{v} > 0$. *In an equilibrium where* J_i *is constant with* $K_i \geq 2$ *on* $[0, \overline{b}]$ *and the following two conditions hold,*

(1)
$$\frac{\sum_{j \in J_i} \left(\Phi(\underline{r}_j, \underline{v}) - \Phi(-\overline{r}_j, \underline{v}) \right)}{K_i - 1} < \Phi(\overline{r}_i, \overline{v}_i) - \Phi(-\underline{r}_i, \overline{v}_i);$$

(2) *there are* $\lambda_j > 0$ *with* $\sum_{j \in J_i} \lambda_j = 1$ *such that*
$$F_j(v) \leq \lambda_j (K_i - 1) \frac{\Phi(-\underline{r}_i, \overline{v}_i)}{\Phi(-\overline{r}_j, v)} \text{ for all } j \in J_i \text{ and all } v \in [\underline{v}, \overline{v}_j];$$

player i *never places a positive bid, i.e.,* $b_i(v) = 0$ *for all* $v \in [\underline{v}, \overline{v}_i]$.

Remark 1. Propositions 1 and 2 can be applied to rule out participation of several contestants in a recursive manner.

The two key assumptions, 1 and 2, imply an expanded version of the familiar inequality comparing the marginal winning probability and the required one. Only in this case, by definition, the latter involves the weighted sum of B_i and L_i. The weights, in general, depend on the probability of winning and hence on the intricate details of the equilibrium, so to derive sufficient conditions for complete drop out, we use extreme values for the weights. For simplicity, the condition is broken into two, and the second one has an independent interpretation. Assumption 2, the first-order stochastic dominance, implies the players believe that the valuations of i's rivals are not very likely to be low, thereby guaranteeing that bid cuts by i will not increase the marginal benefit of winning. This is easy to see in case all the contestants are risk-neutral, as then the marginal benefit of winning is proportional to the marginal winning probability while the marginal cost is constant. By assumption 2, the marginal probability of winning at lower bids is lower than the marginal probability of winning at the top, implying player i can not do better by reducing the bid.

It is clear from prop. 2 that group composition affects one's behavior in a contest with at least three participants. Below are just two easy illustrations.

Example 3 (Weak stay out)**.** Risk-neutral contestants are either *strong* or *weak*, with upper valuations $\overline{v}_s$ and $\overline{v}_w$ respectively, such that $\overline{v}_s > \overline{v}_w$; and lower valuation, $\underline{v} = 0$. Valuations are uniformly distributed. Then, according to prop. 2, weak contestants participate only when the number of strong contestants is smaller than a measure of the relative advantage of the latter, $\frac{\overline{v}_s}{\overline{v}_s - \overline{v}_w}$. Otherwise, all weak contestants shy away from competition by bidding zero regardless of their valuation.

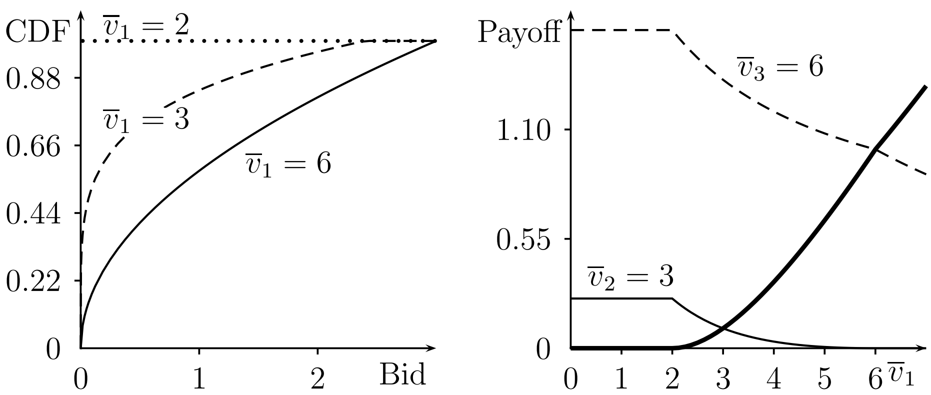

Fig. 2: Complete drop-out. Left: bid density of the first player facing the same two rivals ($\overline{v}_2 = 3$, $\overline{v}_3 = 6$) in three different contests. Right: the equilibrium payoff of players as $\overline{v}_1$ varies from 0 to 7.

Example 4 (Stronger are more active)**.** Two risk-neutral players face a different rival in different contests, see fig. 2. As the upper support of the distribution of the rival grows (in the eyes of the others), his reluctance to participate gives way to an increasingly aggressive behavior in the equilibrium.

4. Discontinuous bidding

Recall that in a standard all-pay auction with incomplete information,[12] there is always an equilibrium in continuous strategies, and in some cases it is the only one.

[12]See, e.g., Krishna and Morgan (1997); Amann and Leininger (1996), Fibich et al. (2006).

Lemma 4. ***(Amann and Leininger, 1996)*** *In a contest with two participants an equilibrium strategy of each is continuous.*

Hence to generate discontinuous behavior in an equilibrium one has to consider contests with at least three players.[13]

Lemma 5. *In a symmetric model, $u_i = u$, $F_i = F$, with all f_i continuous and uniformly bounded above zero on the common support, $[\underline{v}, \overline{v}]$, $\underline{v} > 0$, there is a unique equilibrium in continuous strategies and it is symmetric.*

However, when the symmetry is broken, in particular, if risk attitudes of the contestants differ, the continuous equilibrium might be destroyed.[14]

Proposition 3 (Discontinuous Bidding)**.** *There are at least four contestants; contestants 1 and 2 have identical preferences and distributions of valuations. All the distributions have common support, $[\underline{v}, \overline{v}]$, with $\underline{v} > 0$. Fix the risk-preferences of contestants 3 and 4, if contestants 1 and 2 are sufficiently risk averse, then in any equilibrium at least one contestant uses a discontinuous strategy.*

The discontinuity is "at the bottom." As a contestant becomes more risk averse, he should never place positive bids that bring him the victory with probability below certain threshold. Indeed, bidding at the top is the premium for the full insurance against losing. However, when a contestant places low bids, this analogy breaks down due to the all-pay feature of the auction. A bid still can be seen as a "premium" that the contestant must pay but now the agent only receives probabilistic insurance,[15] with low bids delivering the prize with low probability. A sufficiently risk-averse contestant with low value should get full insurance by bidding zero: the foregone expected gain is his premium in this case. Provided the support of the distribution of the maximal rival bid is an interval, this prescription is consistent with the

[13]However, an anonymous referee pointed that his experimental data set is consistent with discontinuous bidding even in two-player tournaments.

[14]Asymmetry in ex-ante valuations can also generate discontinuity, however, this might not be robust. Assume the players are risk neutral, take a "weak" bidder 1 with distribution of values $F_1(v) = v$ for $v \in [0, 1]$ and $(N-1)$ "strong" bidders, $N > 2$, with the distribution $F_j(v) = v^{1+\alpha}$ for $v \in [0, 1]$, $j \neq 1$ and $\alpha > 0$, then it is possible to show that in any equilibrium where players $j \neq 1$ use the same strategy, player 1 will never place a strictly positive bid below $b_j\left(\left(\frac{N-2}{N-1}\right)^{\frac{1}{\alpha}}\right)$. The argument fails, however, if the lowest valuation is strictly positive.

[15]See, e.g., Kahneman and Tversky (1979, p. 269) for the definition.

equilibrium behavior of the rivals. Hence increasing risk aversion of the agent creates a tension leading to the discontinuity: a mass of low valuation types drops out from the contest, high valuation types place high bids, but no types place low bids. Consequently, this gives rise to *bid bifurcation*, where only bids at zero or bids above a threshold are observed for some players. Such an example, where the equilibrium was computed numerically, is in fig. 3.

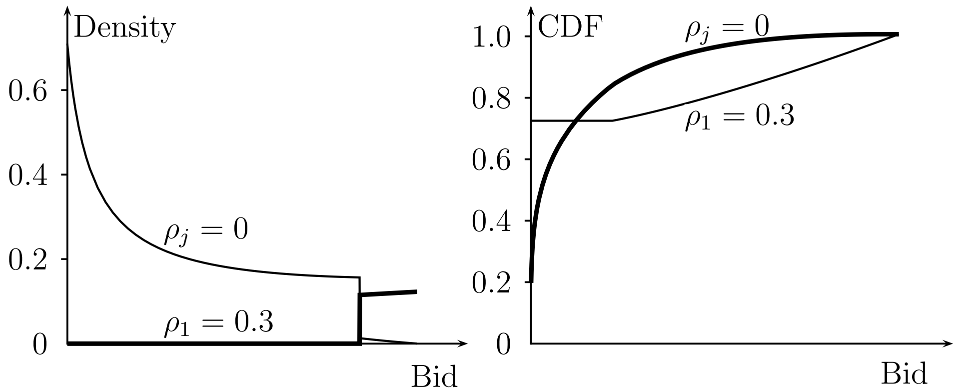

Fig. 3: 4 risk-neutral and 2 CARA competitors with $\rho = 0.3$. Valuations are uniform on $[1/250, 4]$. Note the flat part of the CDF of the risk-averse player 1, corresponding to a 'jump' in his bidding function. Also his CDF indicates an 'atom' at zero.

It is important to remember though that if all agents become *equally* more risk-averse despite the tension outlined above, the bidding will remain continuous in a symmetric equilibrium, thus, ruling out bifurcation of effort. The same conclusion is true in the two-bidder case.

Moreover, the "all-or-nothing" (one-gap) discontinuous equilibrium is ruled out in a symmetric contest altogether. Finally, notice that only asymmetry with respect risk-preferences is required for the discontinuity result, prop. 3 allows for symmetric distribution of valuations (see ft. 14 for an example with different distributions).

Lemma 6. *In a symmetric model, $u_i = u$, $F_i = F$, with all f_i continuous and uniformly bounded above zero on the common support, $[\underline{v}, \overline{v}]$, $\underline{v} > 0$, there is no asymmetric equilibrium where bidding strategies have a finite number of discontinuities.*

Remark 2. Assume the bidding strategy of player 1 in an equilibrium has an infinite number of jumps (discontinuities). Since any equilibrium bid belongs to an interval $[0, \overline{v}]$, which is finite, there should be an infinite subsequence of gaps, with the gap size (or the range of bids never picked in the equilibrium by this player) converging to zero.

Given the range of values, $\overline{v} - \underline{v}$, is bounded too, there should be a point, v_0 with an infinite number of gaps in any of its neighborhoods. Studying existence of such equilibria is beyond the scope of this paper.

Recall that in the first-price auction, for example, increase in risk-aversion of a player *always* raises his bids. In a contest, we find, bids at the top go up as well, however, the low valuation players bid zero to avoid the irreversible investment.

Finally, notice that the *aggregate* distribution of bids should not have gaps in equilibrium. The bifurcation (discontinuity) result highlights the importance of looking at the *individual* behavior in actual contests.

5. Aggressive Bidding

Here we offer two reasons for a player to be more aggressive in a contest: risk aversion and stronger ex-ante distribution of valuations. The results hold for two or more participants.

Recall, prop. 2 shows that a *sufficiently* risk-averse rival chooses a range of bids at the top that are never used by his less risk-averse competitor and prop. 3 implies that in some equilibria a low range of bids will be foregone by a sufficiently risk-averse. The same force pushes the more risk-averse to choose low *strictly positive* effort less often and high effort more often as compared to their less risk-averse rivals in prop. 4, which describes equilibria where all contestants bid in the same interval. This result is "in the same spirit" as prop. 2 by Fibich et al. (2006), who compare behavior of more and less risk-averse players in two different symmetric equilibria, though in our case the contestants with distinct risk attitudes face each other in the same game.

Proposition 4 (Aggressiveness of the Risk-Averse)**.** *Assume valuations share a common support, $[\underline{v}, \overline{v}]$, contestants i and j bid continuously, and contestant i is strictly more risk averse than j. Then there exist a neighbourhood of zero, $N_0 = (0, \varepsilon)$, and that of the upper bid, $N_{\overline{b}} = (\overline{b} - \delta, \overline{b})$, (with $0 < \varepsilon, \delta < \overline{b}$) such that the bid distributions are ordered there, $G_i(b) < G_j(b), \quad b \in N_0 \cup N_{\overline{b}}$.*

Remark 3. So far we have been silent about the likelihood of a more risk-averse agent to choose zero effort. While the probability of him picking low positive effort declines with risk-aversion, he might start choosing zero more often as a result. We leave this as a conjecture: though prop. 3 is consistent with this implication, we can not use the system of differential equations stemming from the optimality conditions 2.1 to describe the equilibrium at zero, as the bid density is not differentiable there.

We conclude our analysis by demonstrating that contestants who are perceived as being "more willing to win" ex-ante are also expected to be more aggressive, i.e., bid higher in the sense of first-order stochastic dominance.[16]

Proposition 5 (Aggressiveness of the Favorite)**.** *Assume all contestants have identical thrice differentiable utility functions, their valuations have a common support,* $[\underline{v}, \overline{v}]$*, and the set of active bidders* $J(b)$ *is constant for all* $b \in [0, \overline{b}]$*. If* $F_j(v) < F_i(v)$ *for all* $v \in (\underline{v}, \overline{v})$*, then* $G_j(b) < G_i(b)$ *for all* $b \in (0, \overline{b})$*.*

6. Conclusions

We extend existing models of incomplete information all-pay auctions by allowing for more than two heterogenous bidders who can be risk-averse. This generalization allows us to rationalize two important phenomena. First, in a traditional winner-take-all contest "underdogs" (viewed as weak by their rivals) might be discouraged from participating; second, relatively more risk-averse agents might choose "all-or-nothing" strategies in such contests.

Although not conclusive, there is some empirical support for both. First, Gneezy et al. (2003) show that women's performance is higher in single-sex than in co-ed pool of contestants, see also Niederle and Vesterlund (2007). Second, Muller and Schotter (2007) and Noussair and Silver (2006) show that a subset of subjects in their all-pay auctions experiments display discontinuous bidding behavior.[17]

Key insight used in the derivation of both results is the *participation test*: if an equilibrium strategy prescribes contestant i to place a strictly positive bid b_0 then the beliefs of his rivals must be consistent, i.e., i's bidding density has to be positive at b_0. The density for an active player can be calculated from the primitives of the model, given equilibrium winning probability. Although the latter depends on the particular equilibrium, one can use the test to formulate *sufficient conditions for non-participation* by taking the appropriate bounds. We also conjecture that the same method can be applied to obtain similar

[16]See Lebrun (1999, cor. 3–*i*, p. 132), Maskin and Riley (2000, prop. 3.3–*ii*, p. 424) and Milgrom (2004, thm 4.25) for a similar exercise for the first-price auction.

[17]For a critical analysis of the approach, see Hoerisch and Kirchkamp (2008).

results for an auction with M homogeneous prizes and $M+2$ or more heterogeneous participants.

Our drop-out and discontinuity results (e.g., illustrations of the first proposition: examples 1–3, as well as prop. 3) can be used to form testable hypotheses. Let us stress again that those should be formulated in terms of individual, and not aggregate behavior.

Previous literature on labor tournaments has established that heterogeneous contests are likely to generate inefficient allocations (Lazear and Rosen, 1981) and that handicaps might reduce the efficiency loss (Schotter and Weigelt, 1992). However, as attention was restricted to just two-player tournaments, participation was not analyzed. Our work complements these contributions by demonstrating that the effect of beliefs and group composition on participation can be non-trivial with at least three distinct agents.

Clearly, we have only looked at some determinants for participation for a *fixed* contest format. If one is to address, say, the question of inducing high participation rates from all qualified groups, which is often stated as an objective of affirmative action programs, then an optimal design framework would be more appropriate. However, we hope that our analysis of the winner-take-all contest provides a first step in that direction.

Appendix A. Proofs

A.1. **The participation test and proofs for section 3.** The following lemma shall be used in the proof of the participation test.

Lemma 7. *For almost all bids,*[18] $b>0$*, the equilibrium first order conditions* (2.1) *can be represented as*

(A.1)

$$\frac{g_i(b)}{G_i(b)} = \begin{cases} \frac{1}{K(b)-1}\left(\sum_{j\in J(b)\setminus\{i\}} S_j(b) - (K(b)-2)S_i(b)\right), & i \in J(b) \\ 0, & \textit{otherwise} \end{cases}$$

where $S_i(b) = \frac{u_i'(\phi_i(b)-b)+\left(\frac{1}{W_i(b)}-1\right)u_i'(-b)}{u_i(\phi_i(b)-b)-u_i(-b)}$.

[18]More precisely, (A.1) holds for all $b>0$ where the inverse bid functions are differentiable and the set of active contestants, $J(b)$, is constant in some neighborhood of b.

Proof. For $i \in J(b)$, differentiating the identity $W_i(b) = \prod_{j \neq i} G_j(b)$, yields

$$\sum_{j \neq i} \frac{g_j(b)}{G_j(b)} = \frac{W_i'(b)}{W_i(b)}, \quad i \in J(b) \tag{A.2}$$

The system (A.2) is linear in the rate of growth of $G_j(b)$, implying

(A.3)

$$\left(\frac{g_j(b)}{G_j(b)}\right)_j = M^{-1} \left(\frac{W_i'(b)}{W_i(b)}\right)_i$$

$$M^{-1} = \frac{1}{K(b)-1} \begin{pmatrix} -(K(b)-2) & 1 & \cdots & 1 \\ 1 & -(K(b)-2) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & -(K(b)-2) \end{pmatrix}$$

For $i \in J(b)$ this proves lemma 7 and for $i \notin J(b)$ we have that $g_i(b) = 0$ provided g_i exists, which is the case for almost every b. ■

Remark 4.

(1) For $i \in J(b)$ with $b > 0$, the first order conditions (2.1) must be satisfied with equality:

$$\frac{W_i'(b)}{W_i(b)} = S_i(b) = \frac{u_i'(\phi_i(b) - b) + \left(\frac{1}{W_i(b)} - 1\right) u_i'(-b)}{u_i(\phi_i(b) - b) - u_i(-b)} > 0 \tag{A.4}$$

(2) Since inverse bidding functions are continuous, $S_i(b)$ is continuous as well.

Corollary 1 (Participation Test)**.** *Consider an equilibrium in which a set $J_i(b)$ of player i's rivals are active at b; they all believe i is so too. Then player i with valuation v does not place bids in a neighborhood of $b > 0$ if*

$$\sum_{j \in J_i(b)} S_j(b) - (K_i(b) - 1) Z_i(b, v) < 0, \quad \textit{where}$$

$$Z_i(b, v) \stackrel{\text{def}}{=} \frac{\exp\left(-\int_{-b}^{v-b} r_i(z)\, dz\right) + W_i^{-1}(b) - 1}{\int_{-b}^{v-b} \exp\left(-\int_{-b}^{y} r_i(z)\, dz\right) dy}. \tag{A.5}$$

Proof of corollary 1. First, definition (A.5) of Z is equivalent to $Z(b) = \frac{u_i'(v-b) + u_i'(-b)\left(\frac{1}{\prod_{j \neq i} G_j(b)} - 1\right)}{u_i(v-b) - u_i(-b)}$, using Pratt's (1964) representation of utilities.

Now assume contrary to the statement that i chooses to bid b in this equilibrium, $b_i(v) = b$, then $Z_i(b,v) = S_i(b)$, which along with the inequality implies (by lemma 7 and the continuity of S_j) that the growth rate of winning probability is negative in a neighborhood of b. Contradiction. ■

Remark 5. The test can be used to rule out participation of player i in the neighbourhood of $b > 0$ for any equilibrium in which $K_i(b) > 1$.

The following lemma shall be used to prove prop. 1.

Lemma 8. *For any $q > 0$, define $b_{q,i} \overset{\text{def}}{=} \inf\{b : W_i(b) \geq q\}$. Fix some $q > 0$ with $b_{q,i} > 0$, let $K \geq 2$ be the number of i's rivals who choose that bid, $K = \#J_i(b_{q,i})$. Contestant i should not bid in the neighborhood of $b_{q,i}$ if*

(1) $\exists\, c > 0$ *such that* $\forall j \neq i$, $W_j(b_{q,j}) \geq c$ *and*
(2) $\frac{1}{K-1}\sum_{j\in J_i(b_{q,i})}\left(\Phi(\underline{r}_j, \phi_j(b_{q,i})) + \frac{1-c}{c}\Phi(-\overline{r}_j, \phi_j(b_{q,i}))\right) <$
$< \Phi(\overline{r}_i, \overline{v}_i) + \frac{1-q}{q}\Phi(-\underline{r}_i, \overline{v}_i)$.

Remark 6. Provided $q > 1$ and $\underline{v} > 0$, condition (2) in lemma 8 can be replaced by

$$\frac{1}{c(K-1)}\sum_{j\in J_i(b_{q,i})}\Phi(-\overline{r}_j, \underline{v}) < \left(\frac{1}{q} - 1\right)\Phi(-\underline{r}_i, \overline{v}_i)$$

Proof. First, using notation of cor. 1,

$$\begin{aligned} Z_i(b,v) &> \frac{1}{\int_{-b}^{\overline{v}_i-b}\exp\left(\int_y^{\overline{v}_i-b}\overline{r}_i\,dz\right)dy} + \frac{W_i^{-1}(b)-1}{\int_{-b}^{\overline{v}_i-b}\exp\left(-\int_{-b}^{y}\underline{r}_i\,dz\right)dy} \\ &= \Phi(\overline{r}_i, \overline{v}_i) + (1/q-1)\,\Phi(-\underline{r}_i, \overline{v}_i). \end{aligned}$$

Next,

$$\begin{aligned} S_j(b) &< \frac{1}{\int_{-b}^{\phi_j(b)-b}\exp\left(\int_y^{\phi_j(b)-b}\underline{r}\,dz\right)dy} + \frac{W_j^{-1}(b)-1}{\int_{-b}^{\phi_j(b)-b}\exp\left(-\int_{-b}^{y}\overline{r}dz\right)dy} \\ &= \Phi(\underline{r}_j, \phi_j(b)) + (1/c-1)\,\Phi(-\overline{r}_j, \phi_j(b)). \end{aligned}$$

The result then follows by cor. 1. ■

Proof of proposition 1. Follows from applying lemma 8, with $c = 1$, $b_q = \overline{b} > 0$ and $q = 1$. ■

Proof of proposition 2. Assume to the contrary contestant i with the highest valuation, $\overline{v}_i$, is bidding some $b > 0$. Given the lowest equilibrium bid is zero, the winning probability of i would have been positive at any $b > 0$, $\Pi_{j\in J_i} G_j(b) = W_i(b) > 0$. Therefore in the view of cor. 1 it is sufficient to verify inequality

$$\sum_{j\in J_i} W_i(b)\, S_j(b) < (K_i - 1) Z_i(b, v)\, W_i(b) \tag{A.6}$$

First, using notation of cor. 1,

$$\begin{aligned} Z_i(b,v)\, W_i(b) &= \frac{W_i(b) + \exp\left(\int_{-b}^{v-b} r_i(z)\, dz\right)(1 - W_i(b))}{\int_{-b}^{v-b} \exp\left(\int_{y}^{v-b} r_i(z)\, dz\right) dy} \\ &> \frac{W_i(b)}{\int_{-b}^{\overline{v}_i - b} \exp\left(\int_{y}^{\overline{v}_i - b} \overline{r}_i\, dz\right) dy} + \frac{1 - W_i(b)}{\int_{-b}^{\overline{v}_i - b} \exp\left(-\int_{-b}^{y} \underline{r}_i\, dz\right) dy} \\ &= W_i(b)\, \Phi(\overline{r}_i, \overline{v}_i) + (1 - W_i(b))\, \Phi(-\underline{r}_i, \overline{v}_i) \end{aligned}$$

Next,

$$\begin{aligned} S_j(b) &< \frac{1}{\int_{-b}^{\underline{v}-b} \exp\left(\int_{y}^{\underline{v}-b} \underline{r}\, dz\right) dy} + \frac{W_j^{-1}(b) - 1}{\int_{-b}^{\underline{v}-b} \exp\left(-\int_{-b}^{y} \overline{r} dz\right) dy} \\ &= \Phi(\underline{r}_j, \underline{v}) + \left(W_j^{-1}(b) - 1\right) \Phi(-\overline{r}_j, \underline{v}) \end{aligned}$$

Note the latter is bounded, as $\underline{v} > 0$. Also, in this equilibrium all the rivals of player i believe that $G_i(b) = 1$, as this bid is chosen by the highest type of player i, hence for any $j \in J_i$ we can write $W_j(b) = \Pi_{k\in J_i\setminus j} G_k$. Then

$$\begin{aligned} \Pi_{k\in J_i} G_k(b)\, S_j(b) &< W_i(b)\, \Phi(\underline{r}_j, \underline{v}) - W_i(b)\, \Phi(-\overline{r}_j, \underline{v}) \\ &\quad + G_j(b)\, \Phi(-\overline{r}_j, \phi_j(b)) \end{aligned}$$

The following condition then implies inequality (A.6),

$$\begin{aligned} &\frac{1}{(K_i - 1)} \sum_{j\in J_i} W_i(b) \left[\Phi\left(\underline{r}_j, \underline{v}\right) - \Phi\left(-\overline{r}_j, \underline{v}\right)\right] + G_j(b)\, \Phi(-\overline{r}_j, \phi_j(b)) \\ < \; & W_i(b) \left[\Phi(\overline{r}_i, \overline{v}_i) - \Phi(-\underline{r}_i, \overline{v}_i)\right] + \Phi(-\underline{r}_i, \overline{v}_i) \end{aligned}$$

The above is implied by the following two inequalities, given at least one of them is strict:

$$\begin{aligned} \frac{1}{(K_i - 1)} \sum_{j\in J_i} \left[\Phi(\underline{r}_j, \underline{v}) - \Phi(-\overline{r}_j, \underline{v})\right] &\leq \Phi(\overline{r}_i, \overline{v}_i) - \Phi(-\underline{r}_i, \overline{v}_i) \\ \frac{1}{(K_i - 1)} \sum_{j\in J_i} G_j(b)\, \Phi(-\overline{r}_j, \phi_j(b)) &\leq \Phi(-\underline{r}_i, \overline{v}_i) \end{aligned}$$

And these conditions are implied by the assumptions 1-2. Note that given we have showed the highest type of player i will never place any

$b > 0$, it is then never optimal for any type of player one to bid above zero in equilibrium. ∎

A.2. **Proofs for section 4.**

Lemma 9. *A symmetric equilibrium strategy of a symmetric contest is strictly increasing on* $[\underline{v}, \overline{v}]$*, satisfies* $b(\underline{v}) = 0$*, solves differential equation*

(A.7)
$$b'(v) = (N-1)\frac{f(v)(F(v))^{N-2}[u(v-b(v)) - u(-b(v))]}{u'(-b(v))\left(1-(F(v))^{N-1}\right) + u'(v-b(v))(F(v))^{N-1}}$$

and so is uniquely determined and is continuous.

Proof. To verify strict monotonicity, by lemma 1, it is sufficient to show the equilibrium distribution of bids does not have an "atom" at zero in this case. Indeed, if it were to have one, a contestant bidding zero with a value strictly above the minimal one would choose, instead, an arbitrarily small bid and discontinuously increase the probability of winning.

The first order conditions in this case imply eq. A.7, with the initial condition determined by lemma 2. This differential equation satisfies Lipschitz condition on $[\underline{v}, \overline{v}]$ and so has a unique solution, which is continuous. ∎

Notation. $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$, $Y_i(b, v) = \frac{u_i'(v_i - b) + u_i'(-b)\left(\frac{1}{\prod_{j\neq i} F_j(v_j)} - 1\right)}{u_i(v_i - b) - u_i(-b)}$.

Lemma 10. *Assume all* f_i *are continuous and uniformly bounded above zero on the common support,* $[\underline{v}, \overline{v}]$ *with* $\underline{v} > 0$*. Then for any* $\overline{b} > 0$ *the system of differential equations, for* $i = 1, \ldots, N$

(A.8)
$$\frac{\partial}{\partial b}\varphi_i(b) = \frac{1}{N-1}\frac{F_i(\varphi_i(b))}{f_i(\varphi_i(b))}\left(\sum_{j\neq i} Y_j(b, \varphi(b)) - (N-2)Y_i(b, \varphi(b))\right)$$

has a unique solution $\varphi = (\varphi)_{i=1}^N$ *on* $[\underline{\beta}, \overline{b}]$ *that satisfies the terminal condition,* $\varphi_i(\overline{b}) = \overline{v}$*, where* $\underline{\beta} \stackrel{\text{def}}{=} \max_i\{b : \varphi_i(b) = \underline{v}\}$.

Proof of lemma 10. There is a neighborhood of $\left(\overline{b}, \varphi(\overline{b})\right)$ such that the system satisfies the Lipschitz condition. Indeed, given the assumptions on f_i, all Y_i are continuous in (b, φ) and bounded for $\varphi > 0$ in a neighborhood of $\left(\overline{b}, \varphi(\overline{b})\right)$. Consequently, the solution $\varphi(b)$ there is locally unique.

Furthermore, as long as $\varphi_i(b) > \underline{v}$ for all i, there is a neighborhood of $(b, \varphi(b))$ where the Lipschitz condition is satisfied. Therefore, $\varphi(b)$

can be further extended by continuity, in a unique way, from $\overline{b}$ to $\underline{\beta}$, which is defined above. ■

Proof of lemma 5. First, lemma 3 implies that the upper bid has to be the same for all contestants in this case, $b_i(\overline{v}) = \overline{b}$. As a consequence, the profile of any continuous equilibrium inverse bid functions must coincide with the unique (up to the upper bound) solution of eq. (A.8) established in lemma 10. Next, the solution of eq. (A.7) in lemma 9, when inverted, is a symmetric solution of eq. (A.8), and hence is its unique solution, being continuous, by lemma 9. But the symmetric solution is fully determined by the initial condition. ■

Proof of lemma 6. Assume, contrary to the statement, that there is at least one player, who uses discontinuous strategy. Among such players consider the highest valuation $v_k \in (\underline{v}, \overline{v})$ for which the bidding function jumps (from $x \geq \underline{v}$ to $y < \overline{v}$), it is well-defined as the number of jumps for each such player is finite. By lemma 3 and lemma 10 given $\overline{b}$ and $\overline{v}$ are strictly positive, the equilibrium strategies in $[v_k, \overline{v}]$ are the inverse $(\varphi_i^{-1})_i$ of the solution of eq. A.8, and, hence, by lemma 5 have to coincide with the symmetric equilibrium. The discontinuity implies the limiting expected payoffs for k are the same, so $\Pi_k(x|v_k) = \Pi_k(y|v_k)$. Equality can not hold for all players, as otherwise there would have been an equilibrium with a common gap for all, which is impossible. Therefore, for at least one contestant i who bids below y, $\Pi_i(x|v_k) < \Pi_i(y|v_k)$. As equilibrium strategies of i and k coincide on $[y, \overline{b}]$, we have $\Pi_i(y|v_k) = \Pi_k(y|v_k)$, a contradiction. Note the last equality is also true if $y = \overline{b}$, so we get the contradiction in this case as well. ■

Proof of proposition 3. Assume all but the first two players bid continuously. Pick a probability of winning $0 < q < 1$. As in lemma 8, let $b_{q,1}$ be the smallest bid associated by player 1 with probability q of winning. No matter what are the preferences of the players, $b_q > 0$. Indeed, if, to the contrary, $b_{q,1} = 0$ then $G_j(0) > q$ for all $j \neq 1$ and moreover since 1 and 2 must choose identical strategies,[19] $G_2(0) = G_1(0) > q > 0$, implying that all players have an atom at zero, which is not possible.

Given $b_{q,1} > 0$, we can apply lemma 8 with $c = q^N$. By remark 6, player 1 (and, so player 2) is not going to pick a bid in some neighbourhood of $b_{q,1}$ if $\frac{1}{N-3} \sum_{j \neq 1,2} \Phi(-\overline{r}_j, \underline{v}) < q^{N-1}(1-q)\Phi(-\underline{r}_1, \overline{v}_1)$. Given Φ is decreasing in the first argument, the last inequality is satisfied if

[19]If strategies are continuous, the system of differential equations given by the first-order conditions has a unique solution for any set of initial conditions with $b > 0$. Moreover, the solution must be the same for players 1 and 2 since they have identical preferences and, by lemma 3, they bid the same at the top.

$\underline{r}_1 > R$ for sufficiently high R, which is bounded given so is the LHS of the inequality (which follows from $\underline{v} > 0$). Clearly, condition 2 of lemma 8 could have been used as well (and it provides an alternative bound for R). ■

A.3. **Proofs for section 5.**

Proof of proposition 4. To prove the claim for b near the top bid, note that by continuity of g_i (lemma 7), it is sufficient to demonstrate that $g_i(\overline{b}) > g_j(\overline{b})$. All the players are bidding the top bid, so, $W_i\left(\overline{b}\right) = 1$ for all i, but then the inequality follows from $S_i(\overline{b}) < S_j\left(\overline{b}\right)$ by lemma 7. The last inequality follows from definition (A.5) in cor. 1, as $r_i(\cdot) > r_j(\cdot)$.

To prove the claim for b near zero, note that by continuity of bidding functions, given $\phi_i(0) = \underline{v}$ and the identity $W_iG_j = W_jG_i$, L'Hôspital's Rule implies $\lim_{b\searrow 0} \frac{G_j(b)}{G_i(b)} = \frac{W_i'(0)}{W_j'(0)}$. Using Pratt's representation of utilities again, by the first order conditions (2.1), we have that $W_i'(0) = \left(\int_0^{\underline{v}} \exp\left(-\int_0^y r_i(z)dz\right) dy\right)^{-1}$ and so $W_i'(0) > W_j'(0)$, as $r_i > r_j$. ■

Proof of proposition 5. First, note that given the set of active bidders is constant on $[0, \overline{b}]$, equilibirium bid densities are fully characterized by lemma 7.

If, contrary to the statement, G_i and G_j intersect at some point in the interior of the support of equilibrium effort levels, $\beta \in (0, \overline{b})$ that is, $G_i(\beta) = G_j(\beta)$, we have $\phi_i(\beta) < \phi_j(\beta)$ by the strict FOSD and the definition of G. It follows then that, using notation of lemma 7, $S_i(\beta) > S_j(\beta)$ and by the characterization of the effort densities from the same lemma, we get $g_i(\beta) < g_j(\beta)$. It implies that at any crossing $b \in (0, \overline{b})$, G_j intersects G_i from below. Finally, using the above argument with $\beta = \overline{b}$, we obtain that $g_i(\overline{b}) = g_j(\overline{b})$ and so G_i and G_j are tangent at $\overline{b}$ but not at any $b \in (0, \overline{b})$.

Hence, it is left to show that in the neigbourhood of the upper bid, $\overline{b}$, $G_j(b) < G_i(b)$.

Since F_j first-order stochastically dominates (FOSD) F_i, we have the weak inequality $f_i(\overline{v}) \leq f_j(\overline{v})$. First consider the case of strict inequality, $f_i(\overline{v}) < f_j(\overline{v})$.

Now, given the differentiability assumptions, W_i'' is well defined on the support[20], so if $G_i(b) = G_j(b)$, $\phi_i(b) = \phi_j(b)$, and $g_i(b) = g_j(b)$ for some $b > 0$ then the following statements are equivalent: $g_i'(b) \geq$

[20]excluding zero and taking $W''(\overline{b})$ to be the left derivative

$g'_j(b) \Leftrightarrow W''_i(b) \leq W''_j(b) \Leftrightarrow \phi'_i(b) \geq \phi'_j(b) \Leftrightarrow f_i(\phi_i(b)) \leq f_j(\phi_j(b))$. These statements follow respectively from the identities[21] listed below:

(1) $$\frac{g'_i G_i - g_i^2}{G_i^2} = \frac{\sum_{k\in J}\left[\frac{W''_k}{W_k} - \left(\frac{W'_k}{W_k}\right)^2\right] - (K-1)\left[\frac{W''_i}{W_i} - \left(\frac{W'_i}{W_i}\right)^2\right]}{K-1}.$$

(2) $$0 = W''_i\,(u_i(\phi_i - b) - u_i(-b)) + (\phi'_i(b) - 1)\,(W'_i\,u'_i(\phi_i - b) + -u''_i(\phi_i - b)) - W'_i\,(u'_i(\phi_i - b) - u'_i(-b)) + (1 - W_i)\,u''_i(-b)$$

(3) $\phi'_i(b) = \frac{g_i(b)}{f_i(\phi_i(b))}$.

It follows $g'_i(\overline{b}) \geq g'_j(\overline{b})$, if and only if, $f_i(\overline{v}) \leq f_j(\overline{v})$. The strict inequality $f_i(\overline{v}) < f_j(\overline{v})$ guarantees the ranking of left derivatives, $g'_i(\overline{b}) > g'_j(\overline{b})$ which means that at the top G_j must intersect G_i from below. We can then conclude then G_i and G_j never intersect at any interior point of the support, that is, G_i is always above G_j.

Finally, we consider the case where $f_i(\overline{v}) = f_j(\overline{v})$. Let $\widetilde{F}_j$ be such that $\widetilde{F}_j(v) < F_i(v)$ for all $v \in (\underline{v}, \overline{v})$ and $\widetilde{f}_j(\overline{v}) > f_i(\overline{v})$. Consider $F_j^\alpha \stackrel{\text{def}}{=} \alpha\widetilde{F}_j + (1-\alpha)F_j$ and also consider the corresponding bid distributions G_j^α and G_i^α, which are continuous in α (as solutions of the system of the differential equations continuous in this parameter). Since in the interior of the support of bids G_i^α is always strictly above G_j^α for any α in the limit as $\alpha \to 0$, G_i is also above G_j and because we previously ruled out tangency, the FOSD ranking must also be strict. ■

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[21]To obtain (1) we differentiate the expression of the growth rate of the bid distribution; as for (2), we differentiate the first-order conditions having previously substituted $\phi_i(b)$ for v_i; and finally for (3), we differentiate the definition of the cumulative distribution of bids, $G_i(b) = F_i(\phi_i(b))$.

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