WELFARE EVALUATION OF POLICIES IN AN OVERLAPPING GENERATIONS GROWTH MODEL

PRELIMINARY DRAFT 1

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Abstract. In the classical growth model, with overlapping generations and continuous time starting at $-\infty$, generically there exist a unique equilibrium in the policy-neighbourhood of any balanced growth equibrium as shown in [5]. Policies are endowment perturbations. In this paper we show that, for any such equilibrium selection, welfare is differentiable, and that the derivative at the balanced growth equilibrium can be computed analytically from the primitives of the model.

1. Introduction

The continuous-time overlapping generations model with exogenous growth is adopted from [2], only we allow for any utility function inducing the same preferences as in that model and that satisfy the homogeneity and concavity requirement in [4].

Notation not explicitly introduced here is in [2], and this includes the definitions of norms (e.g. $\|\cdot\|_{\infty,1}$), notions of differentiability (S^1, sH), etc.

The plan is straightforward: first, the model is introduced and then the main result, theorem 1, establishes differentiability of welfare with respect to endowment perturbations (the value of a public project in coconuts). The last section contains an explicit calculation for the derivative of welfare at a BGE: the perturbation of endowments has both "direct" and "equilibrium" effect. Both effects can be computed analytically.

2. Welfare

2.1. **Utility functions.** The utility function was used till now only in its ordinal aspect; here the cardinal aspect will play a rôle, so we first characterise the cardinal utility functions V, concave and homogeneous as required by [4], which induce the

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same ordinal preferences. This allows in particular to separate risk-aversion (denoted ρ) from intertemporal substitution σ. Then $\rho \geq 0$, $\rho \neq 1$ (for homogeneity), and up to additive and multiplicative constants, if $\sigma \neq 1$: [1]

$$V(c) = \frac{1}{1-\rho}\Big(\int_0^1 e^{-\beta s} c_s^{1-\frac{1}{\sigma}} ds\Big)^{\frac{1-\rho}{1-\frac{1}{\sigma}}}$$

As before we'll omit the non-generic case $\sigma = 1$. As usually, the proof is "obvious".

2.2. RU **normalisation of utility functions.** Since any 2 utility normalisations satisfying ass. 3 of [4] lead to the same result up to a constant factor, we'll fix one of them, dividing by $\frac{V((1+\varepsilon)c^G)-V(c^G)}{\varepsilon}$, which, by homogeneity, equals in the limit $(1-\rho)V(c^G)$. The GRE individual consumption path c^G is used here, rather than e.g. that for the BGE under consideration, in order to keep a common utility scale for the different BGE's of an economy. Thus, utilities are re-scaled such that a 1% increase in GRE consumption yields a utility increase of 1% of a unit. It is then natural to also fix the additive constant such that c^G yields 0 utility, getting this way a canonical utility scale (dimensionless, independent of the measurement units of the goods).

2.3. **Equilibrium utility.** By [5, theorem 1] there exists a locally unique solution to the system described in [2, theorem 1] with strictly positive K_t and all prices (of consumption, output, investment and capital) equal. Substituting in U^* of [2, lemma 4] p and w using [2, thm. 1], and using the notation of [2], yields for the equilibrium utility $(1-\frac{1}{\sigma})U^*_x =$

$$\Big(\int_0^1 e^{-\nu s}\zeta_s ds\Big)^{1-\frac{1}{\sigma}}\Big[e^{\gamma x}\int_0^1 [E_{x+s,s}+(1-\alpha)\varphi_s y_{x+s}]e^{\int_x^{x+s}\mathfrak{r}_u du} ds\Big]^{1-\frac{1}{\sigma}}\Big[\int_0^1 e^{-\eta s+(1-\sigma)\int_x^{x+s}\mathfrak{r}_u du} ds\Big]^{\frac{1}{\sigma}}$$

$$= \Big(\int_0^1 e^{-\nu s}\zeta_s ds\Big)^{1-\frac{1}{\sigma}} e^{(1-\frac{1}{\sigma})\gamma x}\mathcal{N}_x^{1-\frac{1}{\sigma}}\mathcal{D}_x^{\frac{1}{\sigma}}$$

Using then $(1-\rho)V = [(1-\frac{1}{\sigma})U]^{\frac{1-\rho}{1-1/\sigma}}$ we get $(1-\rho)V^*_x = \big[e^{\gamma x}\mathcal{N}_x\mathcal{D}_x^{\frac{1}{\sigma-1}}\int_0^1 e^{-\nu s}\zeta_s ds\big]^{1-\rho}$ (cf. [5, prop. 11.v–vi]). Hence the following:

Definition 1. $\mathcal{V}_x \overset{\text{def}}{=} \frac{\mathcal{N}_x}{(1-\alpha)y^G}[\frac{\mathcal{D}_x}{\Phi(-\eta)}]^{1/(\sigma-1)}$ [cf. 2, cor. 16.g], and $\mathcal{U} \overset{\text{def}}{=} \mathcal{V}\circ\varpi$.

Then normalised equilibrium utility $V_x \overset{\text{def}}{=} [(1-\rho)V_x(c^G)]^{-1}V^*_x - \frac{1}{1-\rho} = \frac{1}{1-\rho}[\mathcal{U}_x^{1-\rho}-1]$.

2.4. **Utility differences.** What we have to sum are the differences w_x of those utilities V_x with those on the baseline, the BGE. Thus, using [2, cor. 13.d–f]:

Lemma 1. *The utility difference* $w_x = \frac{1}{1-\rho}[\mathcal{U}_E^{1-\rho}(x) - \mathcal{U}_0^{1-\rho}(x)]$; *and* $\mathcal{U}_0(x) = \frac{y}{y^G}v(\mathfrak{r})[\frac{\Phi(\mathfrak{r}-\varkappa)}{\Phi(-\eta)}]^{1/(\sigma-1)}$, *where* $y, \mathfrak{r}, \varkappa$ *denote the* BGE *quantities at* $\varpi(0)$, *cf. [5, not. 9.1].*

So the SWF equals $W = \int_{-\infty}^{\infty} e^{\lambda x} w_x dx$, where $\lambda = \nu$ in principle (and then in principle with a multiplicative factor N_0), but is left arbitrary for greater generality.

2.5. **The derivative of welfare.**

Lemma 2. $\mathcal{U}$ *can be added as an additional coordinate of* ϖ, *with the same properties as* $\mathcal{N}$ *and* $\mathcal{B}$, *preserving all statements from the beginning of [5, sect. 11] on.*[2]

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[1] Or, for a representation jointly continuous in ρ and σ, even when crossing 1, let e.g. $V(c) = \frac{1}{1-\rho}\Big[\Big(\frac{1}{\Phi(-\beta)}\int_0^1 e^{-\beta s} c_s^{1-\frac{1}{\sigma}} ds\Big)^{\frac{1-\rho}{1-\frac{1}{\sigma}}} - 1\Big]$.

[2] To illustrate how restrictive the S^1 condition can be, we can't state the same for w_x itself — so we'll need to take a detour —, because there is no analog of lemma [5, 17] for this case: for any $f\colon \mathbb{R}\to\mathbb{R}$, define $\tilde{f}$ on $L_{\infty,1}$ by $\tilde{f}(g) = f\circ g$. Then $\tilde{f}$ is S^1 (or just: has directional derivatives at every constant g) from $L_{\infty,1}$ to itself iff f is affine. Indeed, first differentiability of f follows from the directional derivatives towards constant maps; since the result is clear for affine maps, we can thus subtract from f its tangent at 0, and hence assume $f(0) = f'(0) = 0$. Assume then $f(x_0) \neq 0$ for some $x_0 > 0$, and let $g = \sum_{n>0} n\mathbb{1}_{[n-\frac{1}{n},n]}$. For $\varepsilon = x_0/n_0$, $f(\varepsilon g) = f(x_0)$ on $[n_0 - \frac{1}{n_0}, n_0]$,

Proof. Up to an irrelevant constant factor, $\mathcal{U}$ equals $\mathcal{N}\mathcal{D}^{1/(\sigma-1)}$, and $\mathcal{D}^{1/(\sigma-1)}$ has the same properties as $\mathcal{D}^{-1}$ (C_∞-valued), so all proofs for $\mathcal{B}$ apply to $\mathcal{U}$ as well. ■

Definition 2. Fix $F\colon \mathbb{R} \to \mathbb{R}$ with a bounded, Lipschitz derivative s.t., for some $\eta > 0$ for which [5, thm. 2] guarantees $\varpi_{\mathcal{U}} \geq \eta$, $F(x) = \frac{x^{1-\rho}}{1-\rho}$ for $x \geq \eta$.

Define ϖ_w on B by $\varpi_w(E) = (F\circ\mathcal{U})(E) - (F\circ\mathcal{U})(0)$ and let $\varpi_W = \int_{-\infty}^{\infty} e^{\lambda x} \varpi_{w,x} dx$.

So $\varpi_w = w$ on the set $\{E \in B \mid \operatorname{ess\,sup}_x \int E^-_{x+s,s} ds \leq \delta_1\}$ of [5, thm. 2], in the vein of [5, rem. 43].

Lemma 3. *For a $\mathbb{R}$-valued measurable f, $h\colon L_1^\lambda \to L_1^\lambda\colon (h(u))(t) = F(f_t + u_t) - F(f_t)$ is sH-differentiable.*[3,4]

Proof. The assumptions on F imply $|F(x+y) - F(x) - F'_x y| \leq K y^2/(1+|y|)$. Thus $|F(x+y) - F(x)| \leq K|y|$, so $h\colon L_1^\lambda \to L_1^\lambda$, and suffices to prove its sH-differentiability at $u = 0$, with derivative $F'_{f_t} u_t$. Since $F'_{f_t} \in L_\infty$, it is indeed in $\mathbb{L}(L_1^\lambda, L_1^\lambda)$ ([5, rem. 14]). Suffices thus to prove the convergence property. By the inequality above, this amounts to show that $\frac{1}{\varepsilon} \int e^{\lambda t} \frac{\varepsilon^2 u_t^2}{1+\varepsilon u_t} dt$ goes to 0 with ε, uniformly for $u \geq 0$ in weakly compact subsets of L_1^λ; or, with $C = \frac{1}{\varepsilon}$ and $v(t) = e^{\lambda t} u_t$, that $\lim_{C\to\infty} \int \frac{v(t)^2}{Ce^{\lambda t} + v(t)} dt = 0$ uniformly for $v \geq 0$ in weakly compact subsets of L_1 — so, convergence to 0 for any sequence (C_n, v_n). On bounded intervals $C_n e^{\lambda t}$ is bounded below, say by $D_n \to \infty$; then the integrand is $\leq v_n^2/D_n$, and the square root of this converges in L_1 to 0, and thus in measure: the integrands converge to 0 in measure on bounded intervals; further, each is bounded by v_n, so they are relatively weakly compact in L_1: together those imply the integrals converge to 0. ■

Remark 1. There can clearly be no continuous differentiability in the operator norm, unless using the L_∞ topology on u's. But cf. lemma 6.

Remark 2. In fact, h is $\mathcal{C}$-differentiable, with $\mathcal{C}$ the class of subsets S of $L_1(e^{\lambda t} dt)$ which are uniformly integrable (i.e., $\lim_{N\to\infty} \sup_{f\in S} \int_{|f|>N} |f_t| e^{\lambda t} dt = 0$), and s.t. $\sup_{f\in S} \int_{|f|<N} f_t^2 e^{\lambda t} dt < \infty$. The second condition is automatic for bounded S; and it simplifies, in presence of the first, to $\sup_{f\in S} \int \min\{1, f_t^2\} e^{\lambda t} dt < \infty$.

Indeed, we only need that $\lim_{C\to\infty} \sup_{f\in S} \int \frac{f_t^2}{C+|f_t|} e^{\lambda t} dt = 0$, by the beginning of the above proof. Let then Σ denote the set of distributions of $|f|$ $\forall f \in S$ under $e^{\lambda t} dt$, and Σ' be the set of measures $x\sigma(dx)$ for $\sigma \in \Sigma$: our condition amounts to $\lim_{C\to\infty} \sup_{\sigma\in\Sigma'} \int \frac{x}{C+x} \sigma(dx) = 0$. Thus there must be some $C > 0$ for which the $\sup$ is finite; equivalently, $\sup_{\sigma\in\Sigma'} \int \min\{x, 1\} \sigma(dx) < \infty$: the $\min\{x, 1\}\sigma(dx)$ are a bounded set of measures on $\mathbb{R}_+$. Then, since the $\frac{x}{C+x} / \min\{x, 1\}$ are a uniformly bounded sequence on $\mathbb{R}_+$ decreasing to 0 uniformly on compact sets (and converging to 1 at ∞), we only need the tightness of the $\min\{x, 1\}\sigma(dx)$ to ensure the result, i.e., we need in addition that $\lim_{N\to\infty} \sup_{\sigma\in\Sigma'} \sigma([N, \infty[) = 0$. Translating back to the original f's proves our claim (and that the conditions are necessary and sufficient).

Lemma 4. *The relatively weakly compact sets in $L_{\infty,1} \cap L_1^\lambda$ are those of L_1^λ which are bounded in $L_{\infty,1}$. Their convex circled hulls have the same properties.*

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so $\|\tilde{f}(\varepsilon g)\|_{\infty,1} \geq |f(x_0)|/n_0 = \varepsilon \frac{|f(x_0)|}{x_0}$, for arbitrarily small ε, contradicting that $\tilde{f}$ has 0 as directional derivative at 0 in the direction g. Thus $f(x) = 0$ for $x \geq 0$, and similarly for $x \leq 0$.

[3] Fréchet-differentiability doesn't hold, even with $f = 0$ and $\lambda = 0$: If $F\colon \mathbb{R} \to \mathbb{R}$ is s.t. $h\colon L_1 \to L_1\colon u \mapsto F \circ u - F(0)$ is Fréchet-differentiable at 0, then F is affine. Indeed, subtracting $F(0)$, we can assume $F(0) = 0$. Let h' be the derivative at 0, and $u_x = x\mathbb{1}_B$ where B is a measurable set of finite measure. Then $\|F(x)\mathbb{1}_B - xh'(\mathbb{1}_B)\|_1 = o(x)$ means $\frac{F(x)}{x}\mathbb{1}_B \xrightarrow[x\to 0]{} h'(\mathbb{1}_B)$ in L_1. Thus F is differentiable at 0. No loss then to subtract its derivative at 0, hence to assume $F(0) = F'(0) = 0$. Then the above shows that h' maps any $\mathbb{1}_B$ to 0, hence $h' = 0$. Fréchet-differentiability implies then $\|F \circ g\|_1 \leq \varepsilon \|g\|_1$ for $\|g\|_1$ sufficiently small. Assume $F(x_0) \neq 0$, and let $g = x_0 \mathbb{1}_{[0,\delta]}$: then $\|F \circ g\|_1 = \delta |F(x_0)|$ while $\|g\|_1 = \delta |x_0|$, contradiction.

[4] h also maps $L_{\infty,1} \cap L_1^\lambda$ to itself, but would not even be Gâteaux-differentiable then, cf. fn. 2.

Proof. The intersection is a Banach space by [5, fn. 3]. Necessity of the condition is obvious. Conversely, for any such set, its closed convex circled hull C in L_1^λ is weakly compact [1, 17.12 p. 159, 17.1 p. 154] in L_1^λ, and is so in $L_{\infty,1}$ ($\sigma(L_{\infty,1}, L_{1,\infty})$) by [5, lemma 31](cf. also [3, prop. 1.i,ii]).

Now $\|f\|_{L_{\infty,1}\cap L_1^\lambda} = \mathsf{max}\{\|f\|_{\infty,1}, \|f\|_1^\lambda\}$ is equivalent to $\|f\mathbb{1}_{\mathbb{R}_-}\|_{\infty,1} + \|f\mathbb{1}_{\mathbb{R}_+}\|_1^\lambda$ for $\lambda > 0$, dually for $\lambda < 0$, and is $\|f\|_1$ for $\lambda = 0$. Indeed, say for $\lambda > 0$, the second norm is clearly weaker, and conversely, $\|f\|_{\infty,1} \leq \|f\mathbb{1}_{\mathbb{R}_-}\|_{\infty,1} + \|f\mathbb{1}_{\mathbb{R}_+}\|_{\infty,1}$ where $\|f\mathbb{1}_{\mathbb{R}_+}\|_{\infty,1} \leq \|f\mathbb{1}_{\mathbb{R}_+}\|_1 \leq \|f\mathbb{1}_{\mathbb{R}_+}\|_1^\lambda$, and $\|f\|_1^\lambda \leq \|f\mathbb{1}_{\mathbb{R}_-}\|_1^\lambda + \|f\mathbb{1}_{\mathbb{R}_+}\|_1^\lambda$ where $\|f\mathbb{1}_{\mathbb{R}_-}\|_1^\lambda \leq \|f\mathbb{1}_{\mathbb{R}_-}\|_1 \leq \|f\mathbb{1}_{\mathbb{R}_-}\|_{\infty,1}$.

Thus weak compactness of C in $L_{\infty,1} \cap L_1^\lambda$ is equivalent to weak compactness of $C\mathbb{1}_{\mathbb{R}_-}$ in $L_{\infty,1}$ for $\lambda > 0$ and in L_1^λ else, and of $C\mathbb{1}_{\mathbb{R}_+}$ in $L_{\infty,1}$ for $\lambda < 0$ and in L_1^λ else. Hence, C, being weakly compact in both $L_{\infty,1}$ and L_1^λ, is so in $L_{\infty,1} \cap L_1^\lambda$. ∎

Lemma 5. *For $E \in B_\varepsilon$ and $\lambda \in \Lambda^\varepsilon$, $\delta E \mapsto \varpi_w(E + \delta E) - \varpi_w(E)$ is sH-differentiable at 0 from $L_{\infty,1} \cap L_1^\lambda$ to L_1^λ. The derivative $\varpi'_w(E)$ equals $\delta E \mapsto [x \curvearrowright F'_{\mathcal{U}_E(x)}(\mathcal{U}'_E(\delta E))(x)]$.*

Proof. Take C convex, circled, and weakly compact in $L_{\infty,1} \cap L_1^\lambda$. C being bounded in $L_{\infty,1}$, $\exists r > 0$ s.t. $E + rC \subseteq B_\varepsilon$; let $G(x) = F(\mathcal{U}_{E+x}) - F(\mathcal{U}_E)$ and $g_\eta(x) = \frac{1}{\eta}(G(\eta x) - G(0)) - \phi(x)$, for $\phi \in \mathbb{L}(L_{\infty,1} \cap L_1^\lambda, L_1^\lambda)$; since uniform convergence in L_1^λ of g_η to 0 on rC implies that on C (just multiplying η by r), we can further assume that $E + C \subseteq B_\varepsilon$. Thus, C being convex and symmetric, $g_\eta(x)$ is well defined for $0 < \eta \leq 1$ and $x \in C$, and so is $f(\eta, x) = \frac{1}{\eta}(\mathcal{U}(E + \eta x) - \mathcal{U}(E))$. Define $f(0, x) = \mathcal{U}'_E(x)$, using $\mathcal{U}'$ from lemma 2.

We claim f is jointly continuous from $[0, 1] \times C$ to L_1^λ, using $\sigma(L_1^\lambda, (L_1^\lambda)^*)$ on C and L_1^λ. Indeed the scalar product $(\eta, x) \mapsto \eta x$ is jointly continuous, as for any vector topology; next, since, C being convex and symmetric, ηx varies in C, which is metrisable, [5, thm. 1.iv] implies the continuity of $(\eta, x) \mapsto \mathcal{U}(E + \eta x) - \mathcal{U}(E)$; finally the continuity of the scalar product implies the result for $\eta > 0$. And for $\eta = 0$, since C is bounded in $L_{\infty,1} \cap L_1^\lambda$, $\|f(\eta, x) - f(0, x)\|_1^\lambda$ converges uniformly to 0 on C, by[5] [5, prop. 1.iii], $\mathcal{U}$ being S^1 by lemma 2. Hence the joint continuity.

Thus $D = f([0, 1] \times C)$ is weakly compact in L_1^λ. So, by lemma 3, $\|\frac{1}{\varepsilon}[F(\mathcal{U}_E + \varepsilon y) - F(\mathcal{U}_E)] - F'_{\mathcal{U}_E} y\|_1^\lambda \to 0$ uniformly for $y \in D$, i.e., for any $\delta > 0$ $\exists \varepsilon_0 > 0$ s.t. this norm is $\leq \delta$ $\forall \varepsilon \leq \varepsilon_0, \forall y \in D$. Choose in particular $y = f(\varepsilon, x)$; we get, $\forall \varepsilon \leq \varepsilon_0, \forall x \in C$, $\|\frac{1}{\varepsilon}[F(\mathcal{U}_{E+\varepsilon x}) - F(\mathcal{U}_E)] - F'_{\mathcal{U}_E} f(\varepsilon, x)\|_1^\lambda \leq \delta$. Thus $\|\frac{1}{\varepsilon}[F(\mathcal{U}_{E+\varepsilon x}) - F(\mathcal{U}_E)] - F'_{\mathcal{U}_E}\mathcal{U}'_E(x)\|_1^\lambda \leq \delta + \|F'_{\mathcal{U}_E}(f(\varepsilon, x) - f(0, x))\|_1^\lambda$. Since F' is bounded and $\|f(\varepsilon, x) - f(0, x)\|_1^\lambda$ converges uniformly to 0 on C, decreasing ε_0 some more will ensure that $\|F'_{\mathcal{U}_E}(f(\varepsilon, x) - f(0, x))\|_1^\lambda \leq \delta$ too. ∎

Remark 3. To use rem. 2, following the lines of the above proof, we have to find which sets are mapped by f to $\mathcal{C}$. It is clear that $\mathcal{C}$ is stable under sums (and when taking closed, convex, circled hulls, and adding all measurable functions that are majorised in absolute value by some function in the set). Denote by f_1 and f_2 the functions corresponding to $\mathcal{U}_1$ and $\mathcal{U}_2$, i.e., to $\mathcal{N}_1, \mathcal{B}_1$ and $\mathcal{N}_2, \mathcal{B}_2$. By lemma 2, $\mathcal{U}_i$ has he same properties as $\mathcal{N}_i$ and $\mathcal{B}_i$. Thus, by [5, thm. 1.iii] (for $\lambda = 0, p = \infty$), f_2 has a bounded image on B, so, by the $\frac{\lambda}{2}, p = 2$ case of [5, thm. 1.iii], the image of any subset S of B which is bounded in $L_{2,1}^{\lambda/2}$ belongs to $\mathcal{C}$. Suffices thus to deal with f_1.

Bounding it using [5, cor. 22], it suffices again to deal separately with each of the 2 terms in this bound. The first is $K\int|\delta E_{x+s,s}|ds$, so we need, with $g_{\delta E}(x) = \int|\delta E_{x+s,s}|ds$, that $\mathsf{lim}_{N\to\infty} \mathsf{sup}_{\delta E\in S} \int_{g_{\delta E}>N} g_{\delta E}(x)e^{\lambda x}dx = 0$ and that $\mathsf{sup}_{\delta E\in S} \int \mathsf{min}\{1, g_{\delta E}^2(x)\}e^{\lambda x}dx < \infty$. For the second term, $\left[\frac{1}{\psi_\varepsilon} \star g_{\delta E}\right](x)g_E(x)$,

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[5] Or: [5, rem. 37 and cor. 21], [5, 23 (thm. 1.ii)] yield, taking the L_1^λ-norm on both sides (using lemma [5, 7.v] and [3, prop. 2]), $\|f(\eta, x) - f(0, x)\|_1^\lambda \leq L\eta\|\frac{1}{\psi_\varepsilon}\|_{1,\infty}^\lambda(1 + \|E\|_{\infty,1})\|x\|_{\infty,1}\|x\|_1^\lambda$.

Lemma 6. *ϖ'_w is a continuous map from B_ε to the space of operators from $L_1^\lambda(\mathbb{R}\times[0,1])$ to $L_1^\lambda(\mathbb{R})$ ($\lambda\in\Lambda^\varepsilon$) with the topology of uniform convergence on weakly compact sets.*

Proof. Assume $E_n\to E_0$ in $L_{\infty,1}$, $E_i\in B_\varepsilon$. Then (continuity of S^1 mappings) $\mathcal{U}_{E_n}\to\mathcal{U}_{E_0}$ does so too, and hence, F' being Lipschitz, so does $g_n=F'_{\mathcal{U}_{E_n}}$. F' being bounded, say by M, this implies $g_n\to g_0$ in $\tau(L_\infty,L_1)$. Further, $\mathcal{U}'_{E_n}\to\mathcal{U}'_{E_0}$ in the L_1^λ operator norm, by [5, def. 3, 7 and 1]; let that distance be ε_n. Let now δE vary in a weakly compact set H of L_1^λ, and, given δE, let $f_n=\mathcal{U}'_{E_n}(\delta E)$. Then f_ng_n equals $F'_{\mathcal{U}_{E_n}}\mathcal{U}'_{E_n}(\delta E)$, and $\|f_ng_n-f_0g_0\|_1^\lambda\le\|g_n(f_n-f_0)\|_1^\lambda+\|f_0(g_n-g_0)\|_1^\lambda$. The first term is $\le M\varepsilon_n$, and the second tends to 0 uniformly over H, since $\mathcal{U}'_{E_0}(H)$ is weakly compact in L_1^λ and $g_n\to g_0$ in $\tau(L_\infty,L_1)$. ■

Theorem 1. *For $E\in B_\varepsilon$ and $\lambda\in\Lambda^\varepsilon$, $\delta E\mapsto\varpi_W(E+\delta E)-\varpi_W(E)$ is sH-differentiable at 0 on $L_{\infty,1}\cap L_1^\lambda$. The derivative $\varpi'_W(E)$ equals $\delta E\mapsto\int e^{\lambda x}(\varpi'_{w,E}(\delta E))_x dx$. ϖ'_W is a continuous map from B_ε to $L_\infty^{-\lambda}(\mathbb{R}\times[0,1])$ $\tau(L_\infty^{-\lambda},L_1^\lambda)$.*

Proof. The first part follows from lemma 5, since $f\mapsto\int\phi_\lambda(f)$ is a continuous linear map on L_1^λ. The continuity part follows then in the same way from lemma 6, $L_\infty^{-\lambda}$ being the dual of L_1^λ. ■

Corollary 1. *For $\lambda\in\Lambda^\varepsilon$, at constant policies $E\in B_\varepsilon$ (i.e., $E_{t,s}$ is independent of t), $\varpi'_W(E)=\int e^{\lambda t}q(s)\delta E_{t,s}dsdt$, with $q(s)=A_\lambda+B_\lambda e^{(\mathfrak{r}-\lambda)s}$ for constants A_λ,B_λ.*

Proof. By lemma 2 and [5, prop. 16], $\varpi'_{\mathcal{U}}$ is of the form $a(\cdot)+b(\cdot)\star\big(e^{\mathfrak{r}s}\epsilon_{-s}(\cdot)\big)$. So, by lemma 5, $\mathcal{U}_E(x)$ being constant, $\varpi'_w(\delta E)(x)$ is of the form $\iint\big(a(x-t)\delta E_{t,s}+b(x-t)e^{\mathfrak{r}s}\delta E_{t+s,s}\big)dtds$. Then, by thm. 1, $\varpi'_W(\delta E)=\iiint e^{\lambda(x-t)}a(x-t)e^{\lambda t}\delta E_{t,s}dtdsdx+\iiint e^{\lambda(x-t)}b(x-t)e^{\lambda t+\mathfrak{r}s}\delta E_{t+s,s}dtdsdx$. Since $\delta E\in L_1^\lambda$, and because of the bounds in [5, prop. 16] on the absolutely continuous part of a,b, Fubini's theorem is applicable to those integrals (treating the ϵ_0 part (of b) separately); integrating then first over x (and substituting x by $z+t$) yields, with $A_\lambda=\int e^{\lambda z}a(z)dz$ and $B_\lambda=\int e^{\lambda z}b(z)dz$, $\varpi'_W(\delta E)=\iint A_\lambda e^{\lambda t}\delta E_{t,s}dtds+\iint B_\lambda e^{(\mathfrak{r}-\lambda)s}e^{\lambda(t+s)}\delta E_{t+s,s}dtds$, $=\iint e^{\lambda t}[A_\lambda+B_\lambda e^{(\mathfrak{r}-\lambda)s}]\delta E_{t,s}dsdt$. ■

3. The derivative of welfare made explicit

For multidimensional policy variations, when E is a non constant function of the second variable function $q(s)$ appearing in representation of the derivative of welfare in theorem 1, determines the the welfare evaluation over the "instantaneous" distribution of endowments across ages, or a policy space in [5]. We see here that it is very easy to evaluate: the derivative of welfare is a Laplace transform, and so is constructed from the Laplace transforms of the elementary building blocks. Hence by replacing convolution products by usual products, and using the final formula (with $z=\lambda$) of [5, prop. 12] for $(\frac{\partial F}{\partial k})^{-1}$, the analytic calculation becomes feasible.

Notation 3.1. Let $\mathcal{L}$ denote the Laplace transform mapping $\bigcap_\Lambda L_1^\lambda$ to real-valued functions on Λ: for a function (g) of one real variable let $(\mathcal{L}(g))(z)=\int e^{zt}g(t)dt$ and for a function (G) of two real variables, let $(\mathcal{L}(E))(z,s)=\int e^{zt}E(t,s)dt$.

All the BGE quantities are *-d.

Proposition 1. *The derivative of welfare at a* BGE *with respect to an endowment perturbation is*

$$\delta W=\iint e^{\lambda t}q(s)\delta E(t,s)dsdt=\int q(s)(\mathcal{L}(E))(\lambda,s)ds,\text{ where}$$

$$q(s) = (\mathcal{U}^*)^{-\rho}(q_0(1 - q_1(s)) + q_2(s)), \qquad q_0 = b_0(1 - \mathcal{L}(g))\frac{1}{R - \lambda}, \tag{1}$$

$$q_1(s) = \mathbb{1}_{s\in[0,1]}\frac{\Phi(\lambda - \varkappa)}{\mathcal{D}^*}e^{s(\mathfrak{r}^* - \lambda)}, \qquad q_2(s) = \mathbb{1}_{s\in[0,1]}X^* e^{s(\mathfrak{r}^* - \lambda)} \tag{2}$$

with $\mathcal{L}(g)$ from [5, cor. 12] and with the constant term b_0 defined as

$$X^* \frac{(R^* - \mathfrak{r}^*)(1 - \alpha)}{k^*}\Big[\frac{\mathcal{N}^*}{(\sigma - 1)\mathcal{D}^*}\overline{\mathcal{L}(\mathbb{k}_{\mathfrak{r}}^{\mathcal{D}})} + \overline{\mathcal{L}(\mathbb{k}_{\mathfrak{r}}^{\mathcal{N}})} + \frac{k^*}{(1 - \alpha)}\overline{\mathcal{L}(\mathbb{k}_y^{\mathcal{N}})}\Big] \tag{3}$$

$$\overline{\mathcal{L}(\mathbb{k}_{\mathfrak{r}}^{\mathcal{D}})} = \frac{1 - \sigma}{\mathfrak{r}^* - \varkappa}[\Phi(-\lambda)e^{\mathfrak{r}^* - \varkappa} - \Phi(\mathfrak{r}^* - \varkappa - \lambda)] \tag{4}$$

$$\overline{\mathcal{L}(\mathbb{k}_{\mathfrak{r}}^{\mathcal{N}})} = (1 - \alpha)y^*[v(\mathfrak{r}^*) - v(\mathfrak{r}^* - \lambda)]/\lambda \tag{5}$$

$$\overline{\mathcal{L}(\mathbb{k}_y^{\mathcal{N}})} = (1 - \alpha)v(\mathfrak{r}^* - \lambda) \tag{6}$$

$$\mathcal{U}^* = X^*\mathcal{N}^*,\ X^* = (\mathcal{D}^*)^{\frac{1}{\sigma - 1}},\ \mathcal{D}^* = \Phi(\mathfrak{r}^* - \varkappa),\ \mathcal{N}^* = (1 - \alpha)y^*v(\mathfrak{r}) \tag{7}$$

Proof.

Step 1. *Using definitions*

By sect. 2.4 the derivative of welfare is just a Laplace transform ($\mathcal{L}$) of w_x: $\delta W = \int e^{\lambda x}\delta w_x(\delta E)dx = (\mathcal{L}(\delta w))(\lambda)$, where w_x is $\frac{1}{1-\rho}\mathcal{U}_x^{1-\rho}$ plus a constant. For a given equilibrium selection ϖ present $\mathcal{U}$ as a product, XZ, where $X\colon E \mapsto (\zeta \circ \mathcal{D} \circ \varpi_{\mathfrak{r}})(E)$ with $\zeta\colon \mathbb{R} \to \mathbb{R}\colon z \mapsto z^{\frac{1}{\sigma - 1}}$, and $Z\colon E \mapsto \big(\mathcal{N} \circ (\varpi_{\mathfrak{r}}, \mathbf{1}, \varpi_y)\big)(E)$. So,

$$\begin{aligned}
\delta\mathcal{U} &= \mathcal{N}^*\delta X + X^*\delta Z\\
\delta X &= \frac{1}{\sigma - 1}\frac{X^*}{\mathcal{D}^*}\Big(\frac{\partial\mathcal{D}}{\partial\mathfrak{r}} \circ \varpi_{\mathfrak{r}}'\Big)(\delta E)\\
\delta Z &= \Big(\frac{\partial\mathcal{N}}{\partial\mathfrak{r}} \circ \varpi_{\mathfrak{r}}' + \frac{\partial\mathcal{N}}{\partial E} + \frac{\partial\mathcal{N}}{\partial y} \circ \varpi_y'\Big)(\delta E)
\end{aligned}$$

Step 2. *Gathering terms of δE and $\delta k = \varpi_k'(\delta E)$ at a* BGE

At a BGE (with $E = 0$, $k_t = k^* > 0$, $\mathfrak{r}_t = \mathfrak{r}^*$, $y_t = y^*$, and hence X^*, $\mathcal{D}^*$, $\mathcal{N}^*$ as defined in the statement) one can represent the derivative of $\mathcal{U}$ as a sum of two convolutions using lemma [5, 26.i] and notation [5, 3.1]:

$$\delta\mathcal{U}(x) = (\bar{\mathbb{k}}_E^{\mathcal{U}} \star \epsilon_x)(x) + (\bar{\mathbb{k}}_k^{\mathcal{U}} \star \varpi_k'(\delta E))(x), \tag{8}$$

where

$$\begin{aligned}
&\epsilon_x(t) = \delta E(t, t - x); \quad \mathbb{k}_E^{\mathcal{U}}(s) = X^*\mathbb{1}_{s\in[0,1]}e^{s\mathfrak{r}^*}\\
&\mathbb{k}_k^{\mathcal{U}}(s) = X^*\frac{(R^* - \mathfrak{r}^*)(1 - \alpha)}{k^*}\Big[\frac{\mathcal{N}^*}{(\sigma - 1)\mathcal{D}^*}\mathbb{k}_{\mathfrak{r}}^{\mathcal{D}}(s) + \mathbb{k}_{\mathfrak{r}}^{\mathcal{N}}(s) + \frac{k^*}{(1 - \alpha)}\mathbb{k}_y^{\mathcal{N}}(s)\Big]
\end{aligned}$$

So, $\mathcal{L}(\delta\mathcal{U}) = \mathcal{L}(\bar{\mathbb{k}}_E^{\mathcal{U}}\star\epsilon_x) + \overline{\mathcal{L}(\mathbb{k}_k^{\mathcal{U}})}\mathcal{L}(\varpi_k'(\delta E))$, and the first summand is $\int q_2(s)(\mathcal{L}(\delta E))(\lambda, s)ds$ with q_2 as defined in the statement.

Step 3. *Decomposing $\mathcal{L}(\varpi_k'(\delta E))$ at a* BGE

This is the most exciting part of the computation, based on the implicit function and Wiener theorems. By the implicit function theorem

$$\varpi_k'(\delta E) = -\Big(\frac{\partial F}{\partial k}^{-1} \circ \frac{\partial F}{\partial E}|_{E=0,k=k^*}\Big)(\delta E)$$

By [5, cor. 12] $\frac{\partial F}{\partial k}$ has as inverse in $\bigcap_\Lambda W^\lambda$ a convolution operator $g - \mathbf{1}$, and

$$-\mathcal{L}\Big(\frac{\partial F}{\partial k}^{-1}\Big) = 1 - (\mathcal{L}(g))(\lambda) = \frac{R - \lambda}{(R - \mathfrak{r}^*)(\mathcal{L}H)(\lambda) + \mathfrak{r}^* - \lambda}$$

for $\Re(z) \in \Lambda$, where Λ is the connected component of 0 in $\mathbb{R}\backslash D$ if $(\mathcal{L}H)(R) = 1$ and in $\mathbb{R}\backslash(D \cup \{R\})$ else and H is from lemma [5, 26]. By lemma [5, 24]

$$\frac{\partial F}{\partial E}(t) = \int \mathbb{1}_{t-x>0} e^{-R(t-x)} \delta i_x dx = \Big(\Bbbk_i^{\Upsilon} \star \{ \int \big[\delta E(x,s) - (\Bbbk_{E,s}^c \star \epsilon_s^1)(x)\big] ds \}\Big)(t), \text{ where}$$

$$\Bbbk_i^{\Upsilon}(s) = \mathbb{1}_{s>0} e^{-Rs}, \quad \epsilon_s^1(t) = \delta E(t,s), \quad \delta_{x,s} \text{ is the unit mass at } (x,s)$$

$$\Bbbk_{E,s}^c(z) = (\mathscr{D}^*)^{-1} \mathbb{1}_{0 \le z+s \le 1} \mathbb{1}_{0 \le s \le 1} e^{s(\mathfrak{r}-\varkappa)-\varkappa z}$$

So the Laplace transform ($\mathcal{L}$) of ϖ_k' at the BGE is

$$\mathcal{L}(\varpi_k'(\delta E)) = (1 - \mathcal{L}(g))\mathcal{L}(\Bbbk_i^{\Upsilon}) \int \big[1 - \mathcal{L}(\Bbbk_{E,s}^c)\big]\mathcal{L}(\epsilon_s^1) ds \tag{9}$$

$$= (1 - \mathcal{L}(g)) \frac{1}{R - \lambda} \int \big[1 - q_1(s)\big](\mathcal{L}(\delta E))(\lambda, s) ds \tag{10}$$

Step 4. *Gathering the terms of* $(\mathcal{L}(\delta E))(\lambda, s)$ *to get* $q(s)$

Now the derivative of welfare is evidently linear in $(\mathcal{L}(\delta E))(\lambda, s)$, and the statement follows by verifying that $b_0 = \overline{\mathcal{L}(\Bbbk_k^{\mathfrak{U}})}$. ■

4. The proof of non-vacuity

We prove here non-vacuity for the assumptions of [4].TO BE COMPLETED !!

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