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THE ACCUMULATION OF WEALTH AND THE CYCLICAL GENERATION OF NEW TECHNOLOGIES: A SEARCH THEORETIC APPROACH*

BY BENJAMIN BENTAL AND DAN PELED[1]

In this paper sustained technological progress results from the feedback between technical change and the accumulation of wealth. The production technology is affected by a productivity factor which ensues from research and development. The research and development process is described as a sequential search problem in which optimal decisions depend on current levels of wealth and technology. The resulting growth path displays invention cycles. A discovery of a significant technological improvement at the end of a "search" phase is followed by periods of growth without search. Eventually sufficient wealth is accumulated, research and development resumes and a new cycle begins.

1. INTRODUCTION

This paper explores the mutual dependence between wealth accumulation and technological progress. In the environment described below, technological progress requires an ever-growing investment in research and development (R&D). The necessary investment can be maintained only when wealth is growing. The growth in wealth, in turn, is generated by successful R&D efforts through their effects on production technologies. The resulting growth path is characterized by invention cycles reflecting varying incentives to engage in R&D as technology progresses.

We model technological improvements as random outcomes of a costly search process over potential "untried" technologies. Search efforts are financed by cumulable capital whose alternative use is in the production of goods. The amount of resources spent on search is endogenous, and increases with the stock of the cumulable resource. Increased search activity is likely to generate improved technologies and consequently more output and wealth which, in turn, will enhance future search activities. However, this process is hampered by diminishing returns of two kinds. First, the production process of goods is characterized by the usual decreasing returns to the cumulable factor. Second, as production technologies improve over time, more resources must be allocated to the search process to find additional technological improvements. Accordingly, in order to sustain growth, wealth must grow sufficiently fast to facilitate the required increased R&D investment. Our model identifies some of the factors which affect the interaction between the search process and the production technology, and analyzes their implications

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for the growth patterns and R&D productivity in a simple version of Diamond's (1965) growth model.

The literature on growth and technological progress has by and large abstracted from considering cumulable resources. Consequently, other features were introduced in order to generate sustained output growth through R&D efforts. One common assumption adopted by many authors is a constant returns-to-scale R&D technology, in which a unit invested in R&D generates a constant technological improvement (or a stationary distribution of possible improvements), which in turn shifts the production technology at a constant rate.[2] Some of the notable examples in this line of research include Jovanovic and Rob (1990), Romer (1990), Grossman and Helpman (1991a), and Aghion and Howitt (1992). While successful in generating sustained growth, constant returns-to-scale R&D technology implies that R&D productivity is constant over time, (Kortum 1994). However, there is substantial evidence, both at the macro and micro levels, suggesting that this implication runs counter to the facts.[3] In order to generate declining R&D productivity, Kortum (1994) has departed from the constant returns-to-scale technology of R&D. In the absence of a cumulable resource, Kortum has to assume some *exogenous* growth in the noncumulable resource to sustain growth.[4] The presence of capital in our model enables us to adopt a decreasing returns-to-scale R&D technology while keeping all growth sources endogenous. As a result, we show that it is possible to sustain an R&D-driven growth path, as long as the declining R&D productivity is matched by a sufficiently rapid growth of the capital stock. This result obtains even though the marginal product of capital in any given technology converges to zero.

The idea of modeling R&D as a sequential search process for better technologies has been used, in various forms, by Evenson and Kislev (1975), Nelson and Winter (1982), Telser (1982), and Muth (1986). In our model, profit maximizing firms search for better technologies each period, choosing when to stop the search, and adopt the best technology uncovered.[5] We assume that firms have free access to the *best* technology employed by any firm in the previous period. This feature implies that a typical trajectory of our economy takes the form of repeated "invention cycles." An invention cycle begins with a search phase during which all firms in the economy search for a technological improvement over the "state of the art" technology. Following the discovery of a significant technological improvement, search stops. Because profits in the production of output are high, resources are allocated to production of goods, rather than to R&D, and output growth is high. As the

[2] A completely different approach which underlies much of the "endogenous growth" literature, assumes some externalities or increasing social returns associated with the accumulation of capital, as in Romer (1986) or Lucas (1988).

[3] Griliches (1988), Coe and Helpman (1993), and Kortum (1994) are just some of the authors who document this fact.

[4] This is because without an exogenous growth of the noncumulable resource, the economy cannot sustain the ever-growing requirements of the R&D sector needed to keep the flow of innovations constant.

[5] In some ways, our model may be interpreted as an aggregate version of the model of Jovanovic and MacDonald (1994). They assume an invariant distribution of new technologies within a single-industry dynamic model of technology diffusion. The key difference that generates sustained long-run growth in our model is the endogeneity of the number of draws taken by a searching firm.

marginal product of capital in production of goods tapers off, R&D investment becomes more and more attractive, and eventually firms renew their efforts to improve the technology, starting a new cycle.[6]

Successive invention cycles, although similar in their phase structure, do not have the same characteristics. In particular, as the economy advances technologically it becomes harder to improve upon the state of the art technology. Consequently, the length of the two phases of the cycle, as well as the total cycle length, vary over time. The length of the search phase increases, the no-search intervals become shorter, but the total cycle length grows as the economy progresses. We show further that growing economies experience along their growth paths a particular kind of a transient steady state, in which technological innovations fail to occur despite continued investment in R&D. In such states the observed productivity of R&D and output growth are very low. These situations become more likely as economies become technologically more advanced. Consequently, the model produces a negative correlation between the level of technological advancement and R&D productivity. This prediction is consistent with the facts presented by Griliches (1988), Kortum (1994) and Coe and Helpman (1993) who document increased investment in R&D combined with a decline in R&D productivity and slower growth rates in the U.S.A. and other developed countries.[7]

In Section 2 we present an overview of the model. Section 3 describes a parameterized example. Typical secular trends displayed by the equilibrium path are analyzed in Sections 4 and 5. In particular, we analyze in Section 4 the conditions under which the model generates (stochastic) sustained growth, while Section 5 demonstrates that R&D productivity necessarily declines along these growth paths, despite the long run profitability of that activity. In Section 6 we show that the model produces Schumpeterian "invention cycles," and examine their properties. We provide simulation results of the model, that illustrate the periodicity and size distributions of technological improvements, and measures of R&D productivity. Section 7 contains some remarks on empirical implications of the model which are not pursued here.

2. AN OVERVIEW OF THE MODEL

2.1. *Consumers.* We consider a simple variant of Diamond's (1965) overlapping generations growth model. In each period t, N identical two-period lived agents appear. Agents are endowed with one unit of labor which is supplied

[6] This characterization is akin to the cyclicality in Jovanovic and Rob (1990), where "extensive" search for major innovations, having fixed returns, is followed by a period of declining returns to "intensive" search, during which the major innovation is refined. The fixed return on the extensive search is a result of an assumption that the distribution of improvements generated by that search is fixed. This assumption is tantamount to a constant returns-to-scale R&D technology, discussed above.

[7] Griliches (1988) interprets the observed continued investment in R&D as evidence against the hypothesis that the slow-down in multi-factor productivity is due to a decline in R&D productivity. Our model shows how conditions that warrant continued or even increasing investment in R&D may be consistent with low R&D productivity and the absence of growth.

(inelastically) when young, and allocate the wage income received between first-period consumption and savings. Savings at time t are invested in a fixed number of firms that will operate and generate random profits at time $t+1$. The returns on savings provide the sole source of second-period consumption, and accordingly the amount saved is determined as a solution to an expected utility maximization problem.

2.2. *Firms.* To operate at time t, firms attract savings at period $t-1$ by issuing equities on time t profits. Each firm generates profits by engaging in two distinct but related activities at time t: (i) at the beginning of period t, the firm may conduct a costly *sequential* search for a new technology, with search costs financed by capital raised from time $t-1$ savers; (ii) The remaining resources (net of search costs) are combined with optimally hired labor input to produce output by implementing the technology it uncovered in its own search activity, or by implementing a "technological fallback option" which is common to all firms operating at time t. We assume that the fallback technology is the best technology that has been used by any firm during the previous period. The firm's profits are distributed to share holders as returns on their savings.

2.3. *Optimal R&D Policy.* Firms conduct their search by taking random draws from an infinitely large population of "untried" technologies. Firms examine random draws from that population sequentially, incurring a fixed sampling cost per draw paid out of their beginning of period resources. There is no time-cost involved in R&D efforts *within* the period in addition to the resource cost.

A technology draw completely reveals its productivity level, and the sampling firm can then decide whether to adopt it or reject it. Adopting a technology means stopping the search, and investing all remaining resources in that technology. Rejecting means taking at least one more draw from the distribution of technologies. In addition to having at hand the most recently sampled technology, each firm can adopt at any point during the current period the generally available technological fallback option, and avoid any further search costs. We assume below that draws from the technologies distribution, by any and all firms, are identically and independently distributed, so that sampling is done "with replacement."[8]

Technology draws encountered during the current period search stage are observable only by the firm that found them, including the one that it ultimately adopts. In addition, capital owned by a firm is assumed to become firm-specific. These are the minimal assumptions that suffice to entice firms to invest in R&D when it is profitable to do so and produce on their own, rather than sell-off to a luckier firm. Otherwise, one runs into the usual free ridership problems associated with the gathering of information.[9]

[8] The assumption that a technology draw which is rejected during the current search period cannot be returned to is made for convenience only. The characterization of optimal search "with recall" is more involved, and has little impact on the behavior of the model and its asymptotic properties.

[9] Telser (1982) also adopts similar assumptions for the same reasons.

A broader interpretation of the objective of the search efforts is that firms attempt to boost their profits with business improvements that are inherently nonpatentable and hence nonproprietary. Under this interpretation, the assumption that there is a single-period lag before potential imitation can occur seems appropriate. Since, in addition, firms' owners in our model are short lived, firms behave in a myopic way, seeking to maximize the current period returns on their capital.

A *search strategy* of a firm at any period specifies the rule by which the firm decides when to stop the process of sequential sampling, given its remaining stock of the resource and the best technology known to the firm at that point in time. The optimal search strategy is characterized by threshold acceptance levels, such that a technology sampled at any stage of the search process is accepted if it exceeds the relevant acceptance threshold, and rejected otherwise. As the amount of resources left declines, the acceptance thresholds decrease, and the firm becomes less fastidious. (See the Appendix for a complete description of the optimal search strategy.)

2.4. *Equilibrium.* We consider a *symmetric* equilibrium, in which all firms are identical at the beginning of each period. This symmetry is motivated by the absence of any long-lasting impact of any action taken by the firm at any period. Consequently, the distribution of returns offered by all firms each period will be the same, and all savers will be fully and equally diversified between all firms. It follows that all firms begin activities at period t with the same amount of capital, and that all firms pursue the same search strategy.

In general, an equilibrium consists of the following objects: the beginning of period amount of resources available to each firm, the probability distribution of the returns on investment, the distribution of the wage rate, and a search strategy. These objects are generally related to one another. The search strategy must maximize the firms' objective given their initial resources and the distribution of wages. The distribution of the returns is induced by that search strategy and the distribution of wages. The distribution of wages is generated by a clearing condition in the labor market. This involves equating the sum of all firm-specific demands for labor to the inelastic labor supply. Each firm-specific demand for labor depends on the realizations of the technology level and amount of resources available to it at the end of the search stage. The distribution of the technology and remaining resources depends on the search strategy. Finally, the amount of resources with which each firm starts at the beginning of the period depends on the distribution of returns offered to savers.

To simplify these complicated inter-dependencies, we adopt some simplifying assumptions. These assumptions will render savings independent of the distribution of returns, and the search strategy independent of the distribution of wages.

3. A PARAMETERIZED ECONOMY

3.1. *Savings.* As described above, the economy is populated by two-period lived overlapping generations. Agents maximize expected utility by allocating their wage income between first-period consumption and saving. They use their savings in order to purchase securities with random returns issued by firms. We start by specifying the agents' preferences and optimal saving decisions.

ASSUMPTION 1. *Log Utility. The utility of an agent born at t from first- and random second-period consumptions, $(c_{1t}, \tilde{c}_{2t})$, is given by:*

$$(3.1) \qquad u(c_{1t}, \tilde{c}_{2t}) = \ln(c_{1t}) + \frac{\beta}{1-\beta} E\{\ln(\tilde{c}_{2t})\}, \quad 0 < \beta < 1.$$

Under Assumption 1, savings are independent of future returns. Specifically, a young agent with time t wage income of w_t saves βw_t, regardless of the distribution of the rates of return on savings. Moreover, if all firms offer identical and independent probability distributions of returns, each agent will fully and equally diversify his investment among them.

3.2. *Production and Profits.* Here we describe the two stages of firms' operations: the search stage for new technologies, and the production stage which follows. Since behavior in the search stage depends on the expected profits to be generated in the following production stage, we start with the specification of the production technology and the maximal profit associated with it.

ASSUMPTION 2. *Cob-Douglas Production Function. The combination of k units of capital and l units of labor with a technology indexed by θ yields output given by $A\theta k^{\lambda} l^{(1-\gamma)}$, $A > 0$, $0 < \gamma < 1$, $\theta \geq 1$.*[10]

Each firm enters the production stage with known levels of its production capital and technology, (k, θ), resulting from its search activity during that period. The only decision left at that stage is the choice of labor input. Taking the wage rate, w, parametrically, the profit-maximizing employment level for a firm is:

$$(3.2a) \qquad l^*(k, \theta, w) = k \cdot \left(A\theta \cdot \frac{1-\gamma}{w}\right)^{1/\gamma}.$$

The resulting output and profits are given, respectively, by:

$$(3.2b) \qquad y(k, \theta, w) = (k \cdot \theta^{1/\gamma}) \cdot A^{1/\gamma} \left(\frac{1-\gamma}{w}\right)^{(1-\gamma)/\gamma}$$

$$(3.2c) \qquad \pi(k, \theta, w) = \gamma(k \cdot \theta^{1/\gamma}) \cdot A^{1/\gamma} \left(\frac{1-\gamma}{w}\right)^{(1-\gamma)/\gamma}$$

The search strategy is chosen by each firm in order to maximize the expected profits in (3.2c). These profits are random as of the beginning of the period, being a

[10] The parameter A turns out to have a distinct role in the stochastic growth process described below.

function of the random results of the firm's search activity, (k, θ), and the equilibrium wage rate which depends on the post-search labor demand of all firms.

3.3. *Search for Technologies.* The number of firms that operate at each period is fixed at I.[11] At the beginning of a period, each of the I firms has at its disposal an amount of capital, q, and a default technology, θ^0. Before getting to the production stage, each firm may decide to search for a better technology, by sequentially drawing from a pool of technologies. This pool is described by a fixed and known cumulative probability distribution of the technological index, H: $[1,\infty] \to [0,1]$. Each draw from this pool involves a fixed cost of α units of capital. After each draw, the firm can decide to stop the search process and turn to the production stage, using either its most recently sampled technology or the default technology. In order to produce, the firm hires labor to be combined with the remaining capital. It pays the equilibrium wage rate prevailing at that period.

In equilibrium, the wage rate depends on the random outcomes of the search stage of *all* firms in the economy. Nevertheless, we assume that in deciding whether to take another technology draw or stop the search, each firm takes the distribution of the wage rate as given. In particular, each firm ignores the impact of its own search strategy on the post-search aggregate labor demand, and hence on (the distribution of) the wage rate.

This exogeneity assumption is formally justifiable only with a large number of firms. Instead, one could specify some non-price-taking behavior in the labor market. However, the main properties of the model would remain unaffected. Those properties depend only on the fact, which we establish next, that the optimal search strategy of each firm is an increasing power function of the resources available to it at the beginning of the period. This characteristic of the optimal search strategy seems to be independent of the degree of competition in the labor market. Alternatively, labor hiring can be modeled in a way that is analogous to the way firms raise capital. This would take the form of up-front hiring of labor, *before* firms begin their search for better technologies. Each firm's labor demand would then be based on the *expected* results of its search, and conditioned on the default technology available to all firms in that period. Equilibrium in the pre-search labor market would still require a specification of the market power of each firm, and we believe that a price-taking behavior in that market would retain the power function nature of the optimal search.

Viewing the distribution of the wage rate exogenously, the *expected* profits of a single firm in (3.2c) can be factored into the random variables specific to it, $(\tilde{k}, \tilde{\theta})$, and the equilibrium wage rate, $\tilde{w}$. Accordingly,

$$(3.3) \qquad E\left\{\pi\left(\tilde{k}, \tilde{\theta}, \tilde{w}\right)\right\} = \gamma A^{1/\gamma} \cdot E\left\{\left(\frac{1-\gamma}{\tilde{w}}\right)^{(1-\gamma)/\gamma}\right\} \cdot E\{\tilde{k} \cdot \tilde{\theta}^{1/\gamma}\}.^{12}$$

[11] It turns out that increasing the number of firms lowers both the risk and the mean return on savers' portfolios. This trade-off could be used to endogenize the number of firms operating in each period. We do not pursue this important extension here.

[12] The factorization of the expectations with respect to the wage rate in (3.3) reflects the assumption that each firm ignores the covariance between equilibrium wage rate and its own labor demand.

Consequently, given its beginning of period capital stock q, and the default technology θ^0, each firm follows a search strategy that maximizes $E\{\tilde{k} \cdot \tilde{\theta}^{1/\gamma}\}$, where $\tilde{k}$ and $\tilde{\theta}$ are, respectively, the amount of remaining capital and the technology, available to it at the end of its current period search. The *optimal search strategy* takes the form of a threshold level of technology, $\theta^*(k)$, so that a firm with remaining k units of capital accepts the technology at its disposal if and only if that technology index equals or exceeds $\theta^*(k)$, (see Section 1 in the Appendix).

3.4. *The Distribution of Potential Technologies.* In order to complete the specification of the model, we have to specify the distribution of technologies over which search is conducted.

ASSUMPTION 3. *The probability of drawing a technology index less than or equal to θ is given by the Pareto distribution, $H(\theta) = 1 - \theta^{-\lambda}$, $\theta \geq 1$, $\lambda > 0$.*

The Pareto distribution was chosen for two reasons. First, we want the distribution to be unbounded from above, to reflect the fact that there is always room for further technological improvements. Second, while the density of a Pareto distribution is declining, capturing the idea that better technologies are less likely to be found, the upper tail declines rather slowly.[13] These properties are needed in order to preserve sufficient incentives for R&D even when the economy operates with high levels of technology. Without them, the economy may converge to a neo-classical no-growth equilibrium, since the prospects of further technological improvement obtainable via a fixed-cost sampling do not suffice to offset the declining marginal physical product of capital (a kind of "technological exhaustion").

On the descriptive side, the Pareto distribution has another appealing aspect, that the *relative* improvement over any technology θ^0, conditioned on an improvement being found, has a distribution which is independent of θ^0.[14] Note, however, that in our model firms are interested in the *absolute* level of the technology they employ, rendering the independence of the distribution of *relative* improvements of the default technology irrelevant. Instead, firms in our model have to *increase* their R&D investment over time in order to progress, because the likelihood of improving upon any given default technology (within a fixed number of draws), *decreases* as a function of that technology. In contrast, other models of R&D and growth assume that a given R&D investment generates a fixed distribution of technological improvements, regardless of the current technology (for instance, Jovanovic and Rob 1990, Romer 1990, Grossman and Helpman 1991, and Aghion and Howitt 1992). Thus, assuming that technological *levels*, rather than *improvements*, are drawn

[13] The expected value of the Pareto distribution with $\lambda > 1$ is $\lambda/(\lambda - 1)$, and higher moments exist only up to the order equal to the largest integer less than or equal to λ. The "fat" tail is also related to the *log-convexity* property of the Pareto distribution (see Bagnoli and Bergstrom 1989).

[14] For instance, doubling the processing power of 386 processors and 486 processors would require the same expected R&D investment under this distribution. Kortum (1994) derives the Pareto distribution from the requirement that relative improvements be memoryless.

from a fixed distribution allows for the possibility of decreasing R&D productivity, a possibility which is assumed away in the aforementioned papers.

3.5. *Lower Bounds on θ^* and Equilibrium Aggregate Output.* To characterize the aggregate output in equilibrium it is necessary to obtain the equilibrium wage rate. Letting (k_i, θ_i) denote the post-search realizations of capital and technology of firm i, we get the equilibrium wage rate, which equates the inelastic labor supply to the aggregate labor demand, as:

$$w = (1-\gamma)A(1/N)^{\gamma}\Big(\sum_i \theta_i^{1/\gamma}k_i\Big)^{\gamma}. \tag{3.4}$$

Substituting the equilibrium wage rate in the individual firm's output, (3.2b), yields aggregate output in equilibrium as:

$$Y \equiv \sum y_i(k_i, \theta_i, w) = AN^{(1-\gamma)}\Big(\sum_i \theta_i^{1/\gamma}k_i\Big)^{\gamma}. \tag{3.5}$$

As a first step towards deriving a lower bound on aggregate output, note that when the labor supply is equally divided among the I firms, aggregate output is lower than its equilibrium level. This is true because in equilibrium labor is allocated among the firms according to their marginal product of labor, which is proportional to $\theta_i k_i^{\gamma}$. Accordingly we have the following result:

LEMMA 3.1. *Post-search aggregate output is bounded below by*:

$$Y \geq \sum_i A\cdot(N/I)^{1-\gamma}\cdot\theta_i k_i^{\gamma}. \tag{3.6}$$ [15]

Recall that each firm searches for better technologies in order to maximize $E\{k_i\theta_i^{1/\gamma}\}$. Accordingly, the right hand side of (3.6) is a linear combination of simple convex transformations of individual firms' objectives during the search stage. We use that fact to derive a limiting lower bound on *expected* pre-search aggregate output given the capital stock available at the beginning of the period. First we state that the limiting optimal search strategy has a power function structure.

[15] Formally, Lemma 3.1 can be proven using Holder's Inequality for any two sequences $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \ldots, n$:

$$\Big|\sum a_i b_i\Big| \leq \Big(\sum |a_i|^p\Big)^{1/p}\cdot\Big(\sum |b_i|^q\Big)^{1/q}, \quad \text{where } 1/p + 1/q = 1,\ p > 1,\ q > 1.$$

To get the inequality in (3.6) we let: $b_i = 1$, $a_i = k_i^{\gamma}\theta_i$, and $1/p = \gamma$.

LEMMA 3.2. *The limiting threshold function associated with the optimal search is given by*:

$$\lim_{k \to \infty} \frac{\theta^*(k)}{k^{1/\lambda}} = \left(\frac{\lambda\gamma}{[\alpha(\lambda\gamma - 1)(\lambda\gamma + 1)]} \right)^{1/\lambda}. \tag{3.7}$$

PROOF. See Lemma A.2, Section 1, in the Appendix.[16]

PROPOSITION 3.1. *With Q units of capital equally divided among I firms searching for better technologies, expected aggregate output is asymptotically bounded from below by a power function of Q. Specifically,*

$$\lim_{Q \to \infty} E\{Y|Q\} \geq \Delta \cdot Q^{\gamma + 1/\lambda}, \tag{3.8}$$

where Δ is a constant, given by:

$$\Delta = A \cdot \left(\frac{\lambda\gamma - 1}{\lambda\gamma - \gamma} \right) \cdot \left(\frac{1}{I\alpha(\lambda\gamma + 1)} \right)^{1/\lambda} \cdot N^{(1-\gamma)}.$$

PROOF. See Section 2 in the Appendix.

4. GROWTH

We can now turn to the equilibrium path of the economy, and its long-term characteristics. In particular, we identify the conditions under which the economy is on an R&D-generated growth path.

As stated above, we assume that the default technology which is available to all firms every period is the best technology used by any of the firms in the previous period.[17] Under this assumption, a period in which a significant technological improvement has been discovered is followed by periods in which this new technology is utilized. The rate of return on capital employed in this technology is initially high, and further R&D efforts are likely to be unwarranted. However, as capital accumulates that rate of return is falling, whereas under certain conditions on the characteristics of the pool of untried technologies the alternative expected return on R&D activities remains constant. Eventually firms find it worthwhile to reallocate resources to R&D. We show that under those same conditions renewed R&D

[16] Notice that (3.7) requires $\lambda\gamma > 1$. Although not needed for our results, the existence of a second moment of the Pareto distribution of technologies requires $\lambda > 2$, thus implying $\gamma > 0.5$. Barro and Sala-i-Martin (1992), and Romer (1987) use high values of the capital share, arguing that the commonly measured capital share of 0.3 is biased downwards because of the difficulty of separating human and physical capital.

[17] It can be shown that the conditions we identify below under which the feedback between R&D and wealth accumulation generates sustained growth are independent of the assumption that the best technology becomes public knowledge with a lag of one period. The same asymptotic behavior is obtained also when all knowledge is eroded away at the end of the period, so that every period all firms start searching from scratch. This observation highlights the fact that it is the increasing riches of the economy which matters for our growth mechanism, rather than the "standing on the giants' shoulders" of previous technological improvements which underlies growth in other models. The cyclical aspects of the growth path do depend on the fallback assumption.

activities eventually result in another significant improvement with probability one, after which R&D activities will again be stopped temporarily. As the economy progresses from one such cycle to the next, the search for new technologies becomes harder. In particular, situations in which search efforts fail to uncover technologies which improve upon the available default technology become more likely. In these situations, which we call *quasi-steady-states*, the economy reverts to the existing default technology by the end of the search process. As investment in search activities grows with the level of the default technology, and as both the likelihood and expected duration of quasi-steady-states grow, any measure of R&D productivity which relates the outcome of the R&D effort to R&D investment must be falling, on average, over time. Section 5 further elaborates on this effect.

To demonstrate these features, consider an economy which possesses at time t a default technology $\theta_t = \theta^0$ and an initial stock of aggregate capital $Q_t = Q^0$. We define the state of the economy at t by $x_t = (Q_t, \theta_t)$. In equilibrium, $\{x_t\}$ is a Markov process with a stationary law of motion which depends only on the current state. In particular, given Q_t, firms decide whether to accept θ_t without search, or invest some resources in an effort to uncover a better technology. The best among the technologies actually employed at t, and the resulting aggregate savings at t, constitute the next period state.

We define two mutually exclusive phases in which an economy can be at any period t, given its beginning-of-period state, (Q_t, θ_t). With capital evenly allocated to each of the I firms, optimal search implies that the default technology is accepted without any search if $\theta^*(Q_t/I) \leq \theta_t$, and we say that the economy is in a *no-search phase* at t. In this case the whole resource base Q_t will be used for production. On the other hand, if $\theta^*(Q_t/I) > \theta_t$, we say that the economy is in a *search phase* at t. In this case, some resources will be spent by the firms trying to improve upon the default technology θ_t, and the aggregate output at t becomes a random variable. In either phase, time t savings are a proportion β of time t wage income, which in turn is a proportion $(1-\gamma)$ of current output, Y_t, (see Section 3). The total capital stock available to the economy at $t+1$ both for investment in search and production is therefore:

$$Q_{t+1} = \beta(1-\gamma)Y_t. \tag{4.1}$$

The dynamic behavior of the economy depends on whether the economy ever resumes search once it entered a no-search phase. In order to analyze this question it is useful to define two curves in the $Q-\theta$ plane.

DEFINITION 1. The acceptable technologies curve (AT) is the locus of technologies which are just acceptable given the aggregate capital Q and a default technology θ^0:

$$\text{(AT)} \qquad \theta = \Phi(Q, \theta^0) \Leftrightarrow \theta = \text{Max}\{\theta^*(Q/I), \theta^0\},$$

where $\theta^*(\cdot)$ is the threshold technology characterizing the optimal search.

Thus, given a default technology and capital stock at the beginning of period $t, (Q_t, \theta_t)$, the economy is in a search phase at t if $\theta_t < \Phi(Q_t, \theta_t)$, and is in a no-search phase if $\theta_t \geq \Phi(Q_t, \theta_t)$.

DEFINITION 2. The steady state curve (ST) is the locus of technologies which, absent search activities, are consistent with the aggregate capital Q remaining constant over time:

$$\text{(ST)} \qquad \theta = \Psi(Q) \Leftrightarrow Q = \beta(1-\gamma)A\theta Q^{\gamma}N^{(1-\gamma)}.$$

Absent search activities, the capital level in the economy grows from t to $t+1$ if $\theta_t > \Psi(Q_t)$, stays invariant if $\theta_t = \Psi(Q_t)$, and decreases if $\theta_t < \Psi(Q_t)$.

The location of these two curves in the $Q-\theta$ plane determines the possible paths the economy may take. In particular, the configuration of the two curves determines whether the economy can have sustained, or even increasing growth, or whether the economy must converge to a stationary no-growth equilibrium. Figure 1 presents two possible configurations of these two curves.

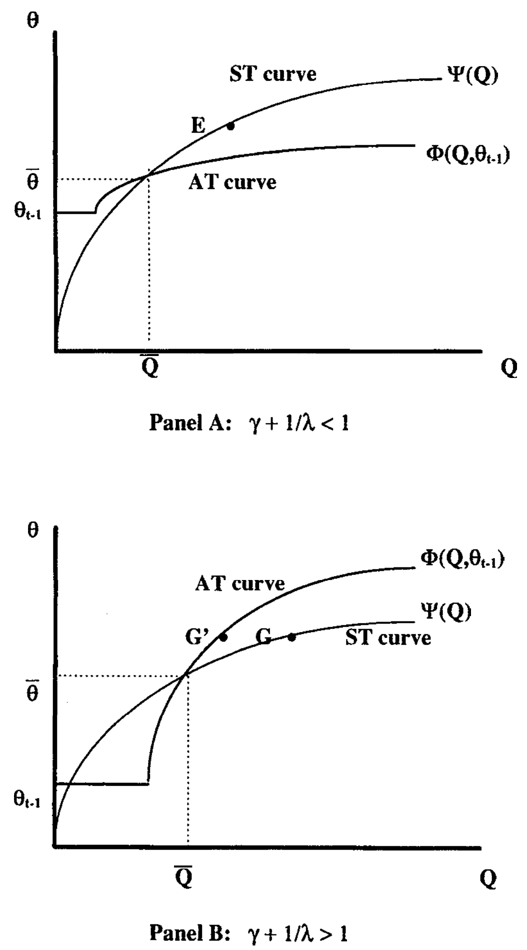

FIGURE 1

In order to see the critical role played by the relative location of the AT and the ST curves, consider an economy at some period t with a state (Q_t, θ_t) given by point E in panel A. This point lies on the ST curve, thus making it a stationary equilibrium absent search. Since E lies *above* the AT curve in that case, θ_t is acceptable given Q_t, and indeed no search takes place. Thus, under the configuration in Panel A, point E corresponds to a stationary no-growth equilibrium. In contrast, under the configuration in Panel B, the state denoted by G, does not constitute a stationary equilibrium. Even though G lies on the ST curve, it is *below* the AT curve, thus indicating that search will be undertaken at that period. This makes the state in the subsequent period random, with possibly better technology and larger capital stock. Neither is a point like G' on the AT curve a stationary equilibrium under the configuration in Panel B, since G' lies above the ST curve, thus prompting a deterministic growth in capital even though no search will be conducted.

As the previous examples indicate, what determines the growth prospects of the economy is whether one of these curves lies permanently above the other for all levels of the capital stock Q above some level $\overline{Q}$. The relative location of the AT and the ST curves turns out to depend in a simple way on the parameters γ and λ. Using Lemma 3.2, Definition 2 of the AT curve immediately leads to the following result.

LEMMA 4.1. *Under the specification of Section* 3, *if* $\gamma + 1/\lambda < 1$, *then for any* θ^0 *there exists a finite* $\overline{Q}$ *such that,* $\Phi(Q, \theta^0) < \Psi(Q)$ *for all* $Q > \overline{Q}$, (*Panel B, Figure* 1). *If* $\gamma + 1/\lambda > 1$ *then for any* θ^0 *there exists a finite* $\overline{Q}$ *such that* $\Psi(Q) < \Phi(Q, \theta^0)$, *for all* $Q > \overline{Q}$, (*Panel A, Figure* 1).

REMARK. For $\gamma + 1/\lambda = 1$ the two curves are parallel for sufficiently large Q. In this case, the ST curve lies below the AT curve for any large enough Q if:

$$\frac{N^{\gamma-1}}{A\beta(1-\gamma)} < \left(\frac{\lambda\gamma}{I\alpha(\lambda\gamma+1)(\lambda\gamma-1)}\right)^{1/\lambda}.$$

In order to characterize the growth path of any given economy, we start with the following observation, which is independent of the location of the AT and ST curves:

PROPOSITION 4.1. *If the economy is in a search phase at* t, *an asymptotic lower bound on expected growth of output from* $t-1$ *to* t *is given by*:

$$(4.2)\quad E\left\{\frac{Y_{t+1}}{Y_t} - 1\right\}$$
$$\geq [\beta(1-\gamma)]^{\gamma+1/\lambda}\left(\frac{1}{\alpha I(\lambda\gamma+1)}\right)^{1/\lambda}\left(\frac{\lambda\gamma-1}{\lambda\gamma-\gamma}\right)AN^{(1-\gamma)}\cdot Y_t^{\gamma+1/\lambda-1} - 1.$$

PROOF. Inequality (4.2) is obtained by using the lower bound on expected output in a search phase, derived in Proposition 3.1, equation (3.8). □

REMARK. When $\gamma + 1/\lambda > 1$, (4.2) implies an increasing expected growth rate for sufficiently large Y_t. Whether the lower bound in (4.2) is positive when $\gamma + 1/\lambda = 1$ depends on the parameters involved. Of course, expected growth may be positive even in cases where the lower bound in (4.2) is not.

According to Proposition 4.1, although output is random in a search phase, the implied expected growth rate is positive as long as search continues, at least for large enough capital stock. We can refine this result, and show that during a search phase the realizations of the growth rates are bounded from below by a bound that approaches zero as the economy grows. The intuitive reason for this result is rather obvious: search is not forced upon firms, and they always have the option to activate the best technology from the previous period. Still, one has to verify that the capital stock with which firms operate at t is (for all practical purposes) at least as large as that of $t-1$ despite the fact that firms maximize profits and not output. These issues are clarified by the following proposition:

PROPOSITION 4.2. *Over an entire search phase, the production capital can drop by no more than α units per firm, and the lowest possible realization of output growth rate approaches zero from below as the capital stock grows.*

PROOF. See Section 3 in the Appendix.

Propositions 4.1 and 4.2 only guarantee positive *expected* output growth rates (and, at the limit, nonnegative realized growth rates as the economy grows), during a search phase. The following proposition specifies the conditions under which growth is positive during the no-search phase. Moreover, it verifies that any no-search phase must be of finite duration, so that the positive expected growth derived in Proposition 4.1 does indeed characterize the economy in the long run.

PROPOSITION 4.3. *If $\gamma + 1/\lambda > 1$ and $Q_t > \bar{Q}$, then the economy grows during any single no-search phase starting at period t or later, and any such phase must be of finite duration.*

PROOF. See Section 3 in the Appendix.

REMARK. If the economy starts with an initial state (Q, θ) such that $Q < \bar{Q}$ and $\theta^*(Q/I) \le \theta \le \theta^*(\bar{Q}/I)$, search activities stop with probability 1 even if $\gamma + 1/\lambda > 1$, and the economy converges to a stationary no-growth equilibrium.[18]

We conclude this discussion by arguing that when $\gamma + 1/\lambda < 1$, growth necessarily stops altogether.

PROPOSITION 4.4. *If $\gamma + 1/\lambda < 1$, then the economy converges to a stationary no-growth equilibrium regardless of initial conditions.*

[18] There are other examples in the endogenous growth literature where economies with a growth potential may fail to grow for lack of sufficient resources, (see Azariadis and Drazen 1990, and Tsiddon 1992).

PROOF. See Section 3 in the Appendix.

Thus we have shown that with Q sufficiently large, $\gamma + 1/\lambda > 1$ suffices to generate positive expected growth rates, while with $\gamma + 1/\lambda < 1$ the economy eventually converges to a no-growth steady-state from any initial condition.

5. R&D PRODUCTIVITY AND INTEREST RATES

In this section we show that with $\gamma + 1/\lambda = 1$, R&D productivity is declining as the economy grows while the interest rate is trendless. As discussed above, and as we show in some detail in the following section, the search process is more likely to fail to improve upon the fallback technology the more advanced that technology is, despite the increasing investment in the search effort. In addition, the number of consecutive periods in which the economy experiences no growth is rising as the technological fallback improves, although its investment in R&D is positive. Therefore, more developed economies are more likely to invest more in search while their realized growth rate remains zero. Accordingly, any measure of R&D productivity is likely to decrease as the economy progresses.

To illustrate this point, we simulated the model 1000 times over 30 periods.[19]

We report two different measures related to R&D productivity. First, we compute the average over all firms of the ratio between the *actual* improvement in the technology θ and the *actual* search investment in R&D. In particular, any technological improvement by any of the searching firms is included in this calculation, even if some other firm found a higher θ. Second, we computed the average economy-wide total factor productivity (TFP) growth. The results are reported in Table 1.

As can be seen, investment in R&D as a fraction of output is trendless. Accordingly, the fact that the average growth rate is almost trendless and positive implies an ever-growing investment in R&D.[20] Nevertheless, the average R&D productivity has a clear downward trend while TFP growth is trendless. The decline in R&D productivity does *not* imply that the *expected* rate of return on R&D is declining. As we argue next, that rate of return (and the interest rate) is bounded below in a growing economy.

The interest rate in the economy can be identified with the expected rate of return on capital, and consequently it depends on the phase of the cycle at which the economy is, as firms seek to maximize the expected return to shareholders'

[19] The economy was started with $Q = 100$, and the initial θ corresponds to the one that would have been acceptable if Q was 94. The other parameters chosen for this simulation (and which are used for all other simulation results reported below) are: $A = 15$, $\beta = 0.5$, $\lambda = 3.333$, $\gamma = 0.7$, $\alpha = 2$, with three firms. This choice was dictated mainly by computational considerations. In particular, the value of λ had to be kept reasonably high to keep the number of search steps (and thereby computation times) at a manageable level. For the same reason we limited the number of periods to 30. Accordingly, the results of the simulations are just indicative of the kind of behavior the model yields, and are by no way to be understood as an attempt to calibrate the model.

[20] There seems to be a certain amount of cyclicality in the growth rate, which reflects the invention cycles discussed in the following section.

TABLE 1
A 30-PERIOD SIMULATION OF THE MODEL*

Period	Average Growth	Std Growth	Average R&D Cost†	Average R&D Prod.	Average TFP Growth	Average θ
0	NA	NA	.0085	NA	NA	2.150
1	.188	.332	.0058	48.297	.0367	2.347
2	.224	.343	.0077	66.486	.0452	2.583
3	.265	.350	.0028	48.603	.0172	2.677
4	.222	.266	.0028	50.014	.0189	2.783
5	.202	.268	.0040	63.814	.0286	2.947
6	.211	.307	.0034	57.229	.0254	3.108
7	.204	.310	.0037	42.495	.0231	3.268
8	.194	.303	.0042	35.686	.0220	3.417
9	.189	.258	.0040	37.510	.0234	3.577
10	.192	.245	.0040	46.313	.0282	3.801
11	.205	.306	.0040	40.752	.0301	4.063
12	.209	.321	.0039	32.477	.0229	4.272
13	.201	.278	.0039	33.662	.0245	4.513
14	.194	.273	.0039	27.935	.0191	4.689
15	.184	.241	.0039	25.170	.0237	4.932
16	.185	.247	.0039	28.556	.0242	5.163
17	.189	.275	.0038	30.809	.0258	5.431
18	.190	.274	.0039	22.335	.0227	5.700
19	.187	.239	.0038	24.093	.0229	5.963
20	.187	.234	.0041	23.248	.0253	6.293
21	.197	.250	.0039	24.473	.0256	6.618
22	.192	.269	.0038	18.323	.0186	6.861
23	.183	.229	.0040	19.775	.0251	7.238
24	.188	.255	.0038	15.976	.0224	7.581
25	.184	.243	.0038	17.293	.0217	7.957
26	.183	.245	.0039	12.351	.0227	8.321
27	.181	.255	.0042	13.160	.0243	8.766
28	.183	.247	.0041	11.681	.0203	9.134
29	.176	.230	.0043	9.263	.0202	9.494
30	.167	.215	.0041	6.808	.0146	9.785

* Explanation of columns: Period (1 through 30); Growth rate from the previous period averaged over 1000 iterations; Standard deviation of the growth rates; R&D costs, measured as the resources spent on search relative to output, averaged over the 1000 iterations. R&D productivity, measured for each iteration as the average ratio of the technological improvement achieved by each firm to its R&D outlays, and averaged over the 1000 iterations; Total factor productivity (TFP) growth, calculated as the fraction of output growth from the previous period not accounted for by growth in factors used in production. The reported number is the average over the 1000 iterations; Average technology level for the period, measured as the average θ actually used in production across all firms and all iterations during the period.

† As a fraction of GDP.

capital. At any period, each firm compares the expected return from employing all its capital in production using the state of the art technology with the expected return from conducting search. Clearly, as long as the marginal product of capital in production is higher, no resources are allocated to search.

With $q = Q/I$ units of capital available to each firm, the expected rate of return on capital when search is undertaken is $E\{\tilde{\pi}/q\}$, where π is given by (3.2c). We can bound this expected rate of return from below, *in equilibrium*, and show that it behaves like $Q^{\gamma+1/\lambda-1}$.[21] When $\gamma + 1/\lambda \geq 1$, this return is nondecreasing, while the marginal product of capital in production is (as long as $\gamma < 1$). Therefore, the expected return on optimally used capital is bounded away from zero under such circumstances. With $\gamma + 1/\lambda = 1$, the lower bound on the expected rate of return in the economy is a constant. In addition, our simulations (with $\gamma + 1/\lambda = 1$) reveal that the maximal realized return on capital, achieved at the first period following a major discovery, is trendless. It follows that the average interest rate in the economy, (or at least its lower bound), is trendless.

6. SCHUMPETERIAN INVENTION CYCLES

6.1. *Search and No-Search Phases.* The fact that the rates of return on resources invested in production and in search for new technologies vary over time and with respect to each other, generates a cyclical pattern in the way resources are allocated to these activities. If the return generated by the state-of-the-art technology is sufficiently high, there is no incentive to engage in R&D. Once it falls, the efforts to find a new technology become worthwhile. Thus, the model generates endogenous Schumpeterian invention cycles from one search phase to the next. An economy which is in a search phase eventually discovers a "major" invention. This discovery is followed by periods without search, during which the invention is utilized. After a while the return on the existing technology falls sufficiently to warrant the beginning of search activities, and that is when a new cycle starts.[22] However, the cycles are not identical along the economy's growth path. This section discusses some of the differences among these cycles with respect to the duration of the search and no-search phases. It also characterizes quasi-steady-states as a particular feature which varies over cycles.

In order to examine cyclical characteristics of the growth process, we have simulated cycles for various initial conditions. To make these simulations comparable, we chose as initial conditions various combinations of initial capital Q^0 and initial default technology θ^0 so that the economy always begins at a search phase. In particular, the default technology was chosen so that each of the I firms accepts that default in the first period of the simulation if it failed to improve it within the first N draws, (that is, $\theta^0 = \theta^*(Q^0/I - N\alpha)$). Thus, each simulated cycle begins with the same absolute level of R&D resources to be spent before the default technology becomes acceptable. Following the initial cycle period, an economy will enter the no-search phase if a sufficiently high θ was found by any of the I firms. Otherwise, if no improvement over θ^0 was found, or the improvement was small enough relative to the resulting Q, it remains in the search phase. In either case, the next period in the cycle begins with the aggregate savings from the previous period

[21] We apply Jensen's Inequality to the RHS of (3.4) to get an upper bound on w in terms of individual firms' expected profits. We then apply it again to the convex function of w in (3.3). With $\gamma > 1/2, 0 < (1-\gamma)/\gamma < 1$, so that:

$$E\{\bar{\pi}/q\} > \gamma A N^{1-\gamma} Q^{\gamma+1/\lambda-1} \cdot \left(\frac{\lambda\gamma}{\alpha I(\lambda\gamma - 1)(\lambda\gamma + 1)} \right)^{1/\lambda}.$$

[22] Schumpeter's (1927) view was that the major innovations, which underlie the "recurring periods of prosperity," are exogenous.

divided equally among the I firms, and the default technology is the best technology utilized by any of the I firms. Once the economy enters the no search phase, we continue the simulation until the capital stock grows to the level where search is resumed, at which period the current cycle ends.

Table 2 below shows the average lengths, in periods, of the search and the no-search phases and the total cycle (standard deviations in brackets) for various specifications of beginning of cycle capital, (Q^0), and the number of draws till the default technology becomes acceptable, (N). In all cases the default technology available at the beginning of the cycle is given by $\theta^*(Q^0/I - N\alpha)$.

The results in Table 2 show that more advanced economies (in the sense of higher initial capital stock and more advanced default technologies) will tend to have longer cycles: 6.017 versus 7.439 periods for $Q^0 = 500$ and $Q^0 = 10000$, respectively, with $N = 40$. However, more striking is the fact that this overall lengthening of the cycle stems from *longer* search phases and *shorter* no-search phases. As our time simulations in the previous section show, the share of R&D investment is without trend, so that longer search spells imply greater R&D investment. Thus, advanced economies spend more time in the search phases, and put more resources into R&D, but with only small technological advancements, or none at all, to show for all their trouble. This pattern is persistent across different values of N.

The effects of N itself are also interesting. Higher values of N correspond to lower levels of the default technology at the beginning of the cycle relative to the economy's wealth. On average, economies with lower default technologies will tend to increase the R&D effort (relative to economies with a higher default), and will tend to find a higher technology level coming out of the search stage at the initial period. Accordingly, we should expect that with a lower default technology, economies should experience *shorter* search phases and *longer* no-search phases. This is what we get in Table 2. We also observe that the overall cycle lengths are somewhat *smaller*. These directions of the impact of the initial default technology

TABLE 2
AVERAGE LENGTH OF CYCLE PHASES*

	Search			No-Search			Total		
	$N=10$	$N=20$	$N=40$	$N=10$	$N=20$	$N=40$	$N=10$	$N=20$	$N=40$
$Q^0 = 500$	2.260 (1.92)	1.471 (1.23)	1.042 (0.42)	4.055 (1.91)	4.319 (1.86)	4.975 (1.61)	6.315 (2.72)	5.790 (2.17)	6.017 (1.64)
$Q^0 = 1000$	3.018 (2.27)	2.413 (2.10)	1.646 (1.69)	3.961 (1.89)	4.017 (1.88)	4.305 (1.80)	6.979 (2.88)	6.430 (2.80)	5.951 (2.40)
$Q^0 = 2000$	3.295 (2.23)	2.991 (2.21)	2.276 (2.01)	3.987 (1.88)	4.010 (1.87)	4.036 (1.85)	7.282 (2.94)	7.001 (2.89)	6.312 (2.74)
$Q^0 = 5000$	3.696 (2.52)	3.553 (2.56)	3.188 (1.83)	3.948 (1.82)	3.785 (1.73)	3.909 (1.84)	7.595 (3.06)	7.501 (3.10)	7.097 (3.06)
$Q^0 = 10000$	3.795 (2.40)	3.721 (2.40)	3.616 (2.40)	3.800 (1.73)	3.785 (1.73)	3.823 (1.73)	7.595 (2.97)	7.506 (2.97)	7.439 (2.94)

*The parameters chosen for this simulation are identical to those used in Table 1: $A = 15$; $\beta = 0.5$; $\lambda = 3.333$; $\gamma = 0.7$; $\alpha = 2$; with three firms.

seem invariant to the initial wealth of the economy, although (as one should expect) the sensitivity to this factor is much greater at lower wealth levels.

Another illustration of the way invention cycles and their phases change as economies develop, is provided by Figure 2. We present the histograms of the length distributions of the search phase and the total cycle length in some of the simulations that produced the averages reported in Table 2. Notice that the duration of the search phase increases with the initial Q^0 (and the associated θ^0), and the

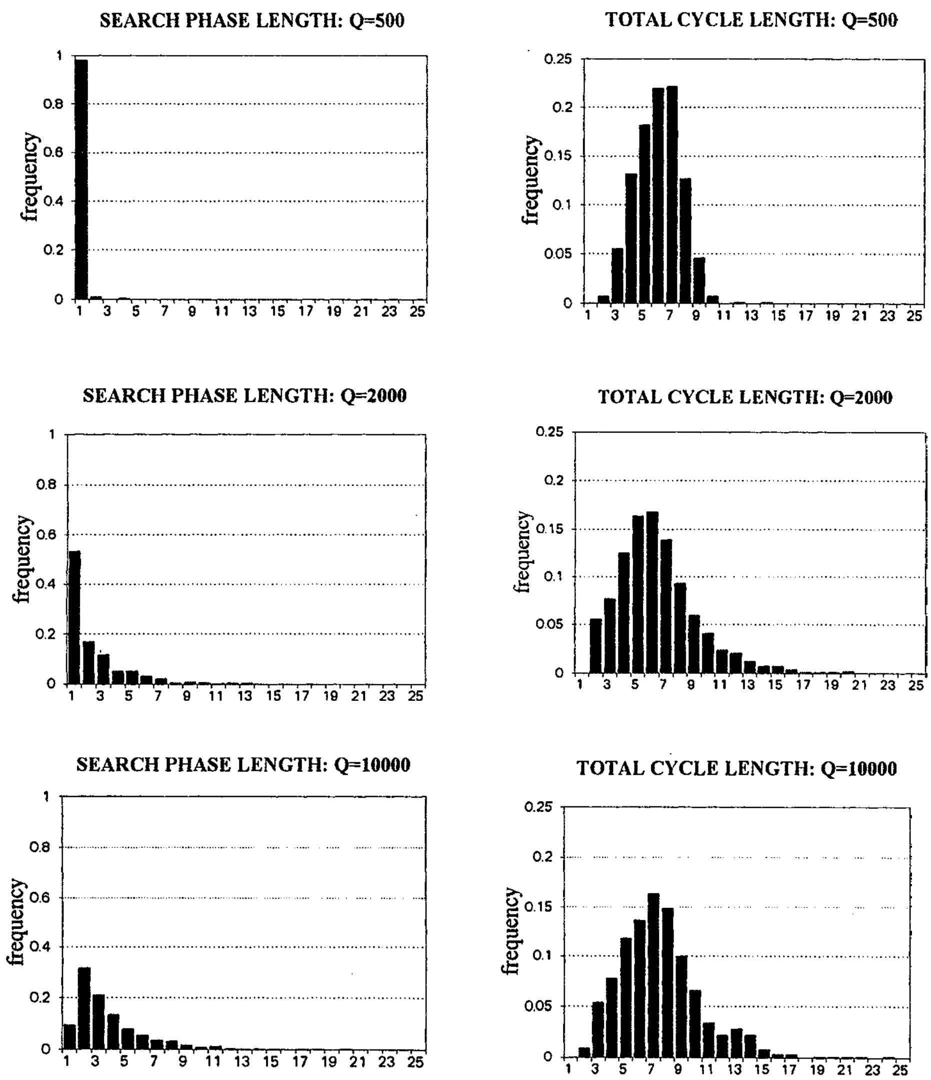

FIGURE 2

CYCLE LENGTH FREQUENCIES

smaller effect in the same direction on the overall cycle length, implying shorter no-search phase.

6.2. *Quasi-Steady-States.* Of particular interest within the search phase is the possibility that despite some positive R&D efforts by the firms no improvement over the default technology is found. These situations, called *quasi-steady-states,* arise when the default technology θ^0, is accepted at two consecutive periods despite some positive R&D effort by the firms, and is activated with approximately the same level of capital. Consequently, the growth rate between such periods is essentially zero, and the economy appears to be repeating its experience.[23]

Formally, a quasi-steady-state for a given aggregate capital in the economy at the beginning of a period Q, is defined by a pair (θ^0, k), where θ^0 is the default technology and k is the ex-search level of remaining capital per firm, which satisfies the following relationships:

(6.1a) $$I\beta(1-\gamma)A\theta^0(N/I)^{1-\gamma}k^\gamma = Q, \quad \text{and}$$

(6.1b) $$\theta^*(k) \leq \theta^0 < \theta^*(k+\alpha).$$

Equation (6.1a) defines k as the per-firm production capital such that when using technology θ^0 the resulting output and savings produce aggregate capital stock of Q for the following period. Equation (6.1b) assures that θ^0 is just acceptable at the firm level with remaining capital stock of k, but is unacceptable if the firm has remaining capital of $k + \alpha$ or more. Thus, in a quasi-steady-state, search stops after each firm finds the default technology acceptable, and saving out of the resulting output restores the given level of capital (with a tolerance of α units per firm).

It follows that a quasi-steady-state is an extreme form of low R&D productivity. Accordingly, it is of some interest to find how likely an economy is to get into such a situation (as a function of its wealth and technological level), and how long such a quasi-steady-state may last.

To get an idea about the likelihood of entering a quasi-steady-state we used the cycle simulations described above with different initial conditions. Table 3 summarizes the percentage of 1000 iterations in which (at least one) quasi-steady-state has occurred during one cycle.[24]

Table 3 shows that quasi-steady-state situations are quite likely, occurring in as much as 62% of the iterations under certain conditions. Moreover, the table clearly shows that wealthier and more advanced economies are more likely to encounter a quasi-steady-state. Finally, the further the economy is from an acceptable technol-

[23] As discussed in Section 4, due to the lumpiness of the search cost, θ^0 is accepted by any firm with k units of capital, $Q^0/I \leq k \leq Q^0/I + \alpha$. Therefore the growth rate need not be exactly zero.

[24] In the period after the discovery of a new technology by any of the firms, the economy may not yet have the capital stock for which that technology is just acceptable. Consequently, we count as QSS only a situation in which the same technology is used in two or more consecutive periods despite positive R&D efforts, and with a capital stock that differs by no more than the drawing resource cost times the number of firms. The latter condition reflects the lumpiness of the optimal R&D investment strategy.

TABLE 3
PROPORTION OF ITERATIONS WITH QUASI-STEADY-STATES*

N	10	20	40
$Q = 500$	0.28	0.111	0.007
$Q = 1000$	0.451	0.320	0.138
$Q = 2000$	0.514	0.447	0.292
$Q = 5000$	0.575	0.530	0.453
$Q = 10000$	0.622	0.599	0.583

* Out of 1000 iterations.

ogy given its wealth, the less likely it is to encounter a quasi-steady-state, since (as argued above) its search during the initial periods is likely to be more intensive. However, Table 3 above does not reveal how *severe* a quasi-steady-state is, only how likely an economy is to run into one. In order to get an idea of the severity of quasi-steady-states as a function of an economy's development, we compute below the probability distribution of the *duration* of these situations.

The number of search steps, n, that each firm undertakes in a quasi-steady-state (θ^0, k) satisfying (6.1a) and (6.1b) is the largest integer which is smaller than $(Q/I - k)/\alpha$. Using the approximation (3.7) and ignoring the integer constraint, we solve (6.1b) as equality for n, to get:

$$(6.2) \quad n = \frac{(\lambda\gamma + 1)(\lambda\gamma - 1)}{\lambda\gamma}(\theta^0)^{\lambda} \cdot \left\{ A\alpha\beta(1-\gamma)\left(\frac{I\alpha(\lambda\gamma+1)(\lambda\gamma-1)}{N\lambda\gamma}\right)^{\gamma-1}(\theta^0)^{\lambda(\gamma+1/\lambda-1)} - 1 \right\}.$$

As can be seen with $\gamma + 1/\lambda \geq 1$, the number of search steps n is an increasing function of θ^0, so that as technology advances, more and more resources are devoted to search in quasi-steady-states.[25]

To complete the analysis, we compute the probability of staying in a quasi-steady-state, given that the economy has entered this state. Staying in the quasi-steady-state corresponds to failing to get a draw higher than θ^0 at the first draw during the search stage, followed by failures to top $\theta^*(Q/I - j\alpha)$ during the following draws, $j = 1, 2, \ldots n-1$. Accordingly, the probability of failing to get out of a quasi-steady-state within one period is:

$$P(Q, \theta^0) = \left\{ \left(1 - (\theta^0)^{-\lambda}\right) \cdot \left(\prod_{j=1}^{n-1} \left[1 - (\theta^*(Q/I - j\alpha))^{-\lambda}\right] \right) \right\}^I.$$

[25] Equation (6.2) also proves that economies with sustained expected growth have quasi-steady-states. With $\gamma + 1/\lambda > 1$, the exponent of θ^0 in the brackets on the RHS of (6.2) is positive for sufficiently large θ^0, and a positive solution for n exists in this case.

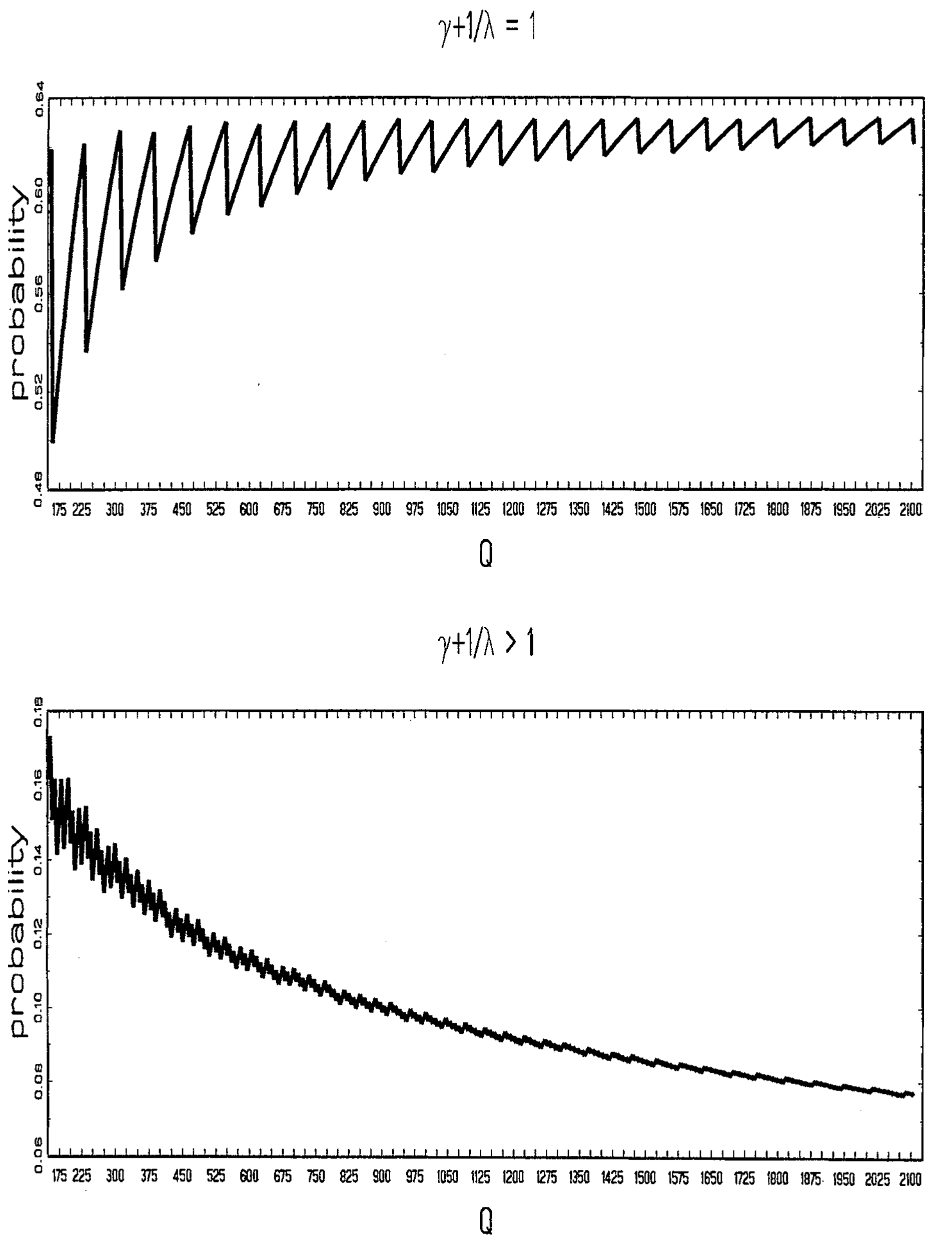

FIGURE 3

PROBABILITY OF STAYING IN QSS

Two conflicting forces determine how the probability of staying in quasi-steady-states varies with Q: as Q increases, so does the number of possible draws prescribed by the optimal search strategy before the corresponding θ^0 becomes acceptable, but the target to be topped increases too. Figure 3 shows that when $\gamma + 1/\lambda = 1$ these forces cancel out for large enough Q, so that the probability of staying in a quasi-steady-state approaches a constant. When $\gamma + 1/\lambda > 1$ this probability is *decreasing*, reflecting the stronger impact of the abundance of resources over the difficulty entailed in further improving on an already advanced technology.[26]

In sum, we have demonstrated that with $\gamma + 1/\lambda = 1$, as economies become technologically more advanced they are more likely to get into quasi-steady-states, tend to spend more time in such states once encountered while allocating more resources to search. All of these results help explain the decline in R&D productivity alongside the sustained growth we obtain in the model, as reported above in Section 4.

7. CONCLUDING REMARKS

The growth path generated by our model possesses some features which resemble the data of the growth process of the industrialized economies. In particular, in our model economies which are richer and more technologically advanced are likely to invest substantially in R&D without actually discovering additional improvements. Thus, any growth that may occur in such economies is attributable mainly to the growth in production factors, and the total factor productivity growth is low. This result is consistent with the decline in total factor productivity growth, the increased investment in R&D and the clear fall of R&D productivity present in the data of most industrialized countries (see Table 1.1 in Grossman and Helpman 1991b).

In addition to the aforementioned macro phenomena which are consistent with our model, our view that R&D is a sequential process has direct testable implications. Many endogenous technological progress models assume that larger investments in R&D generate bigger technological improvements or increase the likelihood of finding an improvement of a given size. These models assume that all the firms engaged in the patent race maintain their R&D investment effort up until the innovation "arrives." None of the competing firms face an increasing pressure to modify their actions as the race progresses. In particular, the R&D expenditures costs are entirely sunk costs. In such models, other things being equal, the bigger spenders on R&D are more likely to be successful. In our description of R&D as an optimal stopping problem, firms respond to their own R&D experience, and can choose to avoid further R&D outlays depending on that experience. One implication of our modeling approach is that among ex-ante identical firms, the correlation between R&D expenditures and innovations should be *negative*. This is so because expensive search efforts indicate repeated R&D failures, while lower R&D expenditures imply an early success in the R&D process. However, *sample averages* of innovation rates and R&D outlays will be *positively* correlated across samples of

[26] The sawtooth nature of the curves in Figure 3 is due to the jumps in the number of search steps that can be taken with a fixed sampling cost (see equation (6.1b)).

firms indexed say, by increasing amounts of initial capital, lower sampling costs, or "better" distributions of untried technologies (which imply, other things being equal, higher search investment). These predictions may help explain the empirical findings of Pakes and Schankerman (1984), who report major differences between intra-industry and inter-industry R&D intensities and growth rates.

Israel Institute of Technology, Israel
Israel Institute of Technology, Israel, and the University of Western Ontario, Canada

APPENDIX

Section 1. *The Optimal Single-Period Search Strategy.* Consider a searching firm having just drawn a technology θ, with k remaining units of capital at its disposal. If θ^0 is the default technology than $z = \text{Max}\{\theta^0, \theta\}$ is the best available technology. Let (k, z) be the state, and $V(k, z)$ be the optimal value function. If the searcher "accepts" (k, z), his reward is $\varphi(k, z)$, where φ increases in both arguments. If (k, z) is rejected and another technology is drawn, the next state will be $(k-\alpha, \tilde{z})$, where $\tilde{z} = \text{Max}\{\tilde{\theta}, \theta^0\}$, $\tilde{\theta}$ being the new realization. If resources do not allow taking another draw, the reward is again $\varphi(k, z)$. Accordingly, the optimal value function $V(k, z)$ must satisfy:

$$\text{(A.1)} \qquad V(k,z) = \begin{cases} \text{Max}\{\varphi(k,z), E\{V(k-\tilde{\alpha}, z) | \theta^0\}\}, & \alpha \leq k, \\ \varphi(k,z), & 0 \leq k < \alpha. \end{cases}$$

LEMMA A1. (i) *For a fixed* $\theta^0 \geq 1$, *and any* $z \geq \theta^0$, *the value function which solves* (A.1) *satisfies*:

$$\text{(A.2)} \qquad V(k,z) = \begin{cases} \varphi(k,z), & \text{if } z \geq \theta^*(k); \quad [\text{"accept"}(k,z)], \\ E\{V(k-\alpha,\tilde{z}) | \theta^0\}, & \text{otherwise}; \quad [\text{"reject"}(k,z)]. \end{cases}$$

(ii) *The values of the threshold function* $\theta^*(\cdot)$ *for* $k = q, q-\alpha, q-2\alpha, \ldots,$ *are the unique solutions to the recursive equation*:

$$\text{(A.3)} \quad \varphi(k, \theta^*(k)) = \text{Max}\Big\{ H[\theta^*(k-\alpha)]\varphi[k-\alpha, \theta^*(k-\alpha)] + \int_{z>\theta^*(k-\alpha)} \varphi(k-\alpha, z)\, dH(z), \varphi(k,\theta^0) \Big\}, \quad k > \alpha,$$

and $\theta^*(k) = \theta^0$ *for* $k \in (0, \alpha]$.

PROOF. Property (i) follows from the single crossing of the two terms in the maximand on the RHS of (A.1), which is due to the fact that $E\{V(k-\alpha, \tilde{z}) | \theta^0\}$ is constant in z, while $\varphi(k, \cdot)$ is continuous and monotonically increasing for $\theta \geq \theta^0$.

For part (ii), note that if $\varphi(k, \theta^0) > E\{V(k-\alpha, \tilde{z}) | \theta^0\}$, then $V(k,z) = \varphi(k, \theta^0)$ for all $z \geq \theta^0$, and we let $\theta^*(k) = \theta^0$. If $\varphi(k, \theta^0) \leq E\{V(k-\alpha, \tilde{z}) | \theta^0\}$, then there exists a unique $z, z \geq \theta^0$, denoted $\theta^*(k)$, for which $\varphi(k, \theta^*(k)) = E\{V(k-\alpha, \tilde{z}) | \theta^0\}$, since $\varphi(k,z) > \varphi(k-\alpha, z)$ for all z. To further characterize $\theta^*(\cdot)$ we proceed inductively, as follows.

Assume that for some integer $n, n \geq 2$, the claim holds for all $k \in (0, n\alpha]$, and consider $k \in (n\alpha, (n+1)\alpha]$. In order to evaluate the second term in the maximand on the RHS of (A.1) we can use the induction hypothesis, since $k - \alpha \in (0, n\alpha]$. This yields, if $n \geq 2$,

$$EV(k-\alpha, \tilde{z}) = H[\theta^*(k-\alpha)] \cdot E\{V(k-2\alpha, \tilde{z}) | \theta_0\}$$

$$+ \int_{z > \theta^*(k-\alpha)} \varphi(k-\alpha, z)\, dH(z).$$

Moreover, by continuity and monotonicity of $\varphi(k-\alpha, \cdot)$, the two branches of (A.2) are equal for $z = \theta^*(k-\alpha)$, so that $E\{V(k-2\alpha, \tilde{z}) | \theta^0\} = \varphi[k-\alpha, \theta^*(k-\alpha)]$, and consequently,

$$E\{V(k-\alpha, \tilde{z}) | \theta^0\} = H[\theta^*(k-\alpha)] \cdot \varphi[k-\alpha, \theta^*(k-\alpha)]$$

$$+ \int_{z > \theta^*(k-\alpha)} \varphi(k-\alpha, z)\, dH(z).$$

Finally, when $n = 1$, $k \in (\alpha, 2\alpha]$, so that $k - \alpha \in (0, \alpha]$, and $\theta^*(k-\alpha) = \theta^0$. In this case $V(k,z) = \varphi(k,z)$ for all $z \geq \theta^0$. Consequently, for $n = 1$ we also have:

$$E\{V(k-\alpha, \tilde{z}) | \theta^0\} = H(\theta^0)\varphi(k-\alpha, \theta^0) + \int_{z > \theta^0} \varphi(k-\alpha, z)\, dH(z)$$

$$= H[\theta^*(k-\alpha)]\varphi[k-\alpha, \theta^*(k-\alpha)]$$

$$+ \int_{z > \theta^*(k-\alpha)} \varphi(k-\alpha, z)\, dH(z).$$

Since $E\{V(k-\alpha, \tilde{z}) | \theta^0\}$ takes on the same form as a function of $\theta^*(k-\alpha)$ for all $k \in (0, n\alpha]$, and is independent of z, Property (ii) of the claim is proven. □

LEMMA A2. *Let* $H(\theta) = 1 - \theta^{-\lambda}$, *and define the following objectives:* $\varphi(k, \theta) = \theta^{1/\gamma}k$, $\varphi_y(k, \theta) = \theta k^\gamma$, $\varphi_l(k, \theta) = \log\{\theta^{1/\gamma}k\}$. *Then the limiting threshold functions as-*

sociated with these objectives are given, respectively,

(A.4) $$\lim_{k\to\infty} \frac{\theta^*(k)}{k^{1/\lambda}} = \left(\frac{\lambda\gamma}{[\alpha(\lambda\gamma-1)(\lambda\gamma+1)]}\right)^{1/\lambda};$$

(A.5) $$\lim_{k\to\infty} \frac{\theta_y^*(k)}{k^{1/\lambda}} = \left(\frac{\lambda}{[\alpha(\lambda-1)(\lambda\gamma+1)]^{1/\lambda}}\right)^{1/\lambda};$$

(A.6) $$\lim_{k\to\infty} \frac{\theta_l^*(k)}{k^{1/\lambda}} = \frac{1}{[\alpha(\lambda\gamma+1)]^{1/\lambda}}.$$

PROOF. Substituting $H(\theta)=1-\theta^{-\lambda}$, and the specified objectives $\varphi(k,\theta)$, $\varphi_y(k,\theta)$, and $\varphi_l(k,\theta)$ into (A.3) yields after rearranging the following recursive equations in $\theta^*(k)$, for $k \ge \alpha$, respectively:

(A.7) $$\theta^*(k) = \text{Max}\left\{\left(1+1/(\lambda\gamma-1)\cdot\theta^*(k-\alpha)^{-\lambda}\right)^{\gamma}\cdot\theta^*(k-\alpha)\cdot(1-\alpha/k)^{\gamma}, \theta^0\right\},$$

(A.8) $$\theta_y^*(k) = \text{Max}\left\{\left(1+(\lambda-1)\cdot\theta_y^*(k-\alpha)^{-\lambda}\right)\cdot\theta_y^*(k-\alpha)\cdot(1-\alpha/k)^{\gamma}, \theta^0\right\},$$

(A.9) $$\theta_l^*(k) = \text{Max}\left\{\exp(\theta_l^*(k-\alpha)^{-\lambda}/\lambda)\cdot\theta_l^*(k-\alpha)\cdot(1-\alpha/k)^{\gamma}, \theta^0\right\},$$

and $\theta^*(k) = \theta_y^*(k) = \theta_l^*(k) \equiv \theta^0$, for $k \le \alpha$.

It can be shown, when $\lambda\gamma > 1$, that the first term in each of the maximands of (A.7)–(A.9) exceeds the second for sufficiently large k. Moreover, the threshold functions are monotonically increasing in k for large enough k, approaching infinity as $k \to \infty$.

For brevity sake, we only demonstrate (A.9), the limiting behavior of $\theta_l^*(k)$. The limits in (A.7) and (A.8) can be established in similar ways. For $k > \alpha$, define $z(k) \equiv \theta_l^*(k)/(k^{1/\lambda})$. We show that $z(k)^{\lambda} \to 1/[\alpha(\lambda\gamma+1)]$, by showing that $z(k) < z(k-\alpha)$ if $z(k-\alpha)^{\lambda} > 1/[\alpha(\lambda\gamma+1)]$, while $z(k) > z(k-\alpha)$ if $z(k-\alpha)^{\lambda} < 1/[\alpha(\lambda\gamma+1)] + o(k)$, where $o(k) \to 0$ as $k \to \infty$.

Using the definition of $z(k)$ we get from (A.9):

$$z(k) = [(k-\alpha)/k]^{\gamma+1/\lambda} z(k-\alpha)\cdot\exp\left(z(k-\alpha)^{-\lambda}(k-\alpha)^{-1}/\lambda\right),$$

or in logarithmic form:

(A.10) $$\ln z(k) - \ln z(k-\alpha) = 1/\left[\lambda\cdot z(k-\alpha)^{\lambda}\cdot(k-\alpha)\right] - (\gamma+1/\lambda)\cdot\ln[k/(k-\alpha)].$$

Now, use the fact that for $x \in (-1,1]$,

$$\ln(1+x) = x - x^2/2 + x^3/3 - \dots,$$

to conclude that $\ln[1+\alpha]/(k-\alpha)[<\alpha/(k-\alpha)$, so that:

$$\text{(A.11)}\quad \ln z(k)-\ln z(k-\alpha)$$
$$>1/\left[\lambda\cdot z(k-\alpha)^{\lambda}\cdot(k-\alpha)\right]-[(\lambda\gamma+1)/\lambda]\cdot[\alpha/(k-\alpha)]$$
$$=1/[\lambda(k-\alpha)]\cdot\left(1/\left[z(k-\alpha)^{\lambda}\right]-\alpha(\lambda\gamma+1)\right).$$

Thus, $\ln z(k)<\ln z(k-\alpha)$ when $z(k-\alpha)^{\lambda}>1/[\alpha(\lambda\gamma+1)]$.

Likewise, use the fact that $\ln(1+x)>x-x^2/2$, for $x=\alpha/(k-\alpha)\in(-1,1]$, to conclude from (A.10) that:

$$\text{(A.12)}\quad \ln z(k)-\ln z(k-\alpha)$$
$$<\frac{1}{\lambda(k-\alpha)}\cdot\left(\frac{1}{z(k-\alpha)^{\lambda}}-\alpha(\lambda\gamma+1)+\frac{\alpha^2}{z(k-\alpha)}\right).$$

It follows that $\ln z(k)>\ln z(k-\alpha)$ when $z(k-\alpha)^{\lambda}>1/[\alpha(\lambda\gamma+1)]+o(k)$. □

Section 2. *Proof of Proposition* 3.1. We first bind from below $E\{\theta k^{\gamma}\}$, where k and θ are the realizations of the end of search capital and technology of a single firm. In order to do that, we use the search strategy of a firm that maximizes expected *log* of profits, denoted $\varphi_l(k,\theta)=\ln(k\theta^{1/\gamma})$. Modifying the firm's objective affects both the threshold technology levels that characterize the firm's optimal search, as well as the distribution of the search outcome. We indicate the use of a different search strategy than the one corresponding to expected profits maximization by explicitly conditioning the expectations on the strategy in use, (e.g. θ_l^* for expected log of profits maximization).

LEMMA A3. $E\{\tilde{\theta}\cdot\tilde{k}^{\gamma}|\theta_l^*\}>\theta_l^*(q)\cdot q^{\gamma}$.

PROOF. The expected value of $\varphi(k,\theta)$ under an optimal search strategy, when the searcher starts with q units of capital and the default technology is $\theta^0\le\theta^*(q)$, is $\varphi(q,\theta^*(q))$, (see part (ii) in Lemma A1). Using the fact that $\varphi_l(k,\theta)=\ln(k\theta^{1/\gamma})$, and Jensen inequality yields:

$$\text{(A.13)}\quad E\left\{\ln(\tilde{\theta}\cdot\tilde{k}^{\gamma})|\theta_l^*\right\}=\gamma E\left\{\ln(\tilde{\theta}^{1/\gamma}\cdot\tilde{k})|\theta_l^*\right\}\ge\ln\left[\theta_l^*(q)\cdot q^{\gamma}\right].$$

Further,

$$\text{(A.14)}\quad E\left\{\tilde{\theta}\cdot\tilde{k}^{\gamma}|\theta_l^*\right\}=E\left\{e^{\ln[\tilde{\theta}\cdot\tilde{k}^{\gamma}]}|\theta_l^*\right\}>e^{E\{\ln[\tilde{\theta}\cdot\tilde{k}^{\gamma}]|\theta_l^*\}}\ge\theta_l^*(q)\cdot q^{\gamma},$$

where the first inequality in (A.14) follows from applying Jensen's inequality, so that $E\{\exp(\tilde{x})\}>\exp(E(\tilde{x}))$ for any nondegenerate random variable $\tilde{x}$, and the second equality in (A.14) follows from (A.13). □

By Lemma A3, equation (3.5) implies that expected aggregate output when all firms use θ_l^*, and where each of them has Q/I units of capital at the beginning of search, is bounded below by:

$$E\{\tilde{Y}|\theta_l^*\} > AN^{(1-\gamma)} \cdot Q^{\gamma} \cdot \theta_l^*(Q/I). \tag{A.15}$$

Next, we utilize the limiting approximation of $\theta_l^*(\cdot)$ for the Pareto distribution, as derived in Lemma A2, to get a limiting bound on expected aggregate output under θ_l^*:

$$E\{\tilde{Y}|\theta_l^*\} > A[\alpha(\lambda\gamma+1)I]^{-1/\lambda} N^{(1-\gamma)} Q^{\gamma+1/\lambda}. \tag{A.16}$$

However, firms' actual search strategy is represented by $\theta^*(\cdot)$, not by $\theta_l^*(\cdot)$. The two results presented below allow us to relate the expected value of $\tilde{\theta} \cdot \tilde{k}^{\gamma}$ under the optimal search strategies for different objectives.

LEMMA A4. *Consider two objective functions φ_1 and φ_2, and their associated optimal threshold functions $\theta_1^*(\cdot)$ and $\theta_2^*(\cdot)$, respectively. Then, $\theta_1^*(k) > \theta_2^*(k)$ for all $k > \alpha$, if φ_2 is a concave nondecreasing transformation of φ_1.*

PROOF. Let $\varphi_2 = \Phi[\varphi_1]$, Φ concave nondecreasing transformation. For $k \in (0, \alpha]$, we have that $\theta_1^*(k) = \theta_2^*(k) = \underline{\theta}$. Assume that $\theta_1^*(k) \geq \theta_2^*(k)$ for all $k \in (0, K]$, $K \geq \alpha$, and consider $k' \in (K, K+\alpha]$. From (A.3), (ignoring θ^0), we have:

$$\varphi_2(k', \theta_2^*(k')) = E \operatorname{Max}\{\varphi_2(k'-\alpha, \theta_2^*(k'-\alpha)), \varphi_2(k'-\alpha, \tilde{\theta})\}.$$

Using the relation $\Phi(\varphi_1) = \varphi_2$, the last equation can be written as:

$$\Phi[\varphi_1(k', \theta_2^*(k'))] = E \operatorname{Max}\{\Phi[\varphi_1(k'-\alpha, \theta_2^*(k'-\alpha))], \Phi[\varphi_1(k'-\alpha, \tilde{\theta})]\},$$

and applying Jensen's inequality, using the concavity of Φ,

$$\Phi[\varphi_1(k', \theta_2^*(k'))] < \Phi[E \operatorname{Max}\{\varphi_1(k'-\alpha, \theta_2^*(k'-\alpha)), \varphi_1(k'-\alpha, \tilde{\theta})\}].$$

By assumption, $\theta_2^*(k'-\alpha) \leq \theta_1^*(k'-\alpha)$, so that with Φ and φ_1 being nondecreasing we can replace $\theta_2^*(k'-\alpha)$ on the right hand side by $\theta_1^*(k'-\alpha)$ to get:

$$\Phi[\varphi_1(k', \theta_2^*(k'))] < \Phi[E \operatorname{Max}\{\varphi_1(k'-\alpha, \theta_1^*(k'-\alpha)), \varphi_1(k'-\alpha, \tilde{\theta})\}].$$

Since Φ is nondecreasing, it follows that:

$$\varphi_1(k', \theta_2^*(k')) < E \operatorname{Max}\{\varphi_1(k'-\alpha, \theta_1^*(k'-\alpha)), \varphi_1(k'-\alpha, \tilde{\theta})\}$$
$$= \varphi_1(k', \theta_1^*(k')).$$

With φ_1 being nondecreasing in θ, we have: $\theta_2^*(k') < \theta_1^*(k')$. □

Since $\theta^*(k)$ corresponds to the objective $k\theta^{1/\gamma}$, while $\theta_l^*(k)$ corresponds to the objective $\ln(k\theta^{1/\gamma})$, Lemma A4 implies that $\theta^*(k) \geq \theta_l^*(k)$.

Next, consider two general search strategies, given by their associated threshold functions $\theta(k)$ and $\theta'(k)$, for a search from a Pareto distribution, with initial pre-search stock of q units of capital.

LEMMA A5. *Let $\theta_1^*(\cdot)$ and $\theta_2^*(\cdot)$ be two threshold functions associated with two search strategies from the same Pareto distribution with the same initial stock of q units of capital, and the same default technology, θ^0. Suppose there exists an $x > 0$, such that $\theta_1^*(k) \leq \theta_2^*(k) < (1+x)\theta_1^*(k)$, for all $k > \alpha$. Then $E\{\tilde{\theta}\cdot\tilde{k}^\gamma | \theta_2^*\} \geq (1+x)^{-\lambda}\cdot E\{\tilde{\theta}\cdot\tilde{k}^\gamma | \theta_1^*\}$.*

PROOF. Without loss of generality assume that θ^0 is low enough so that at least one draw from the distribution H is taken under both strategies. Then, the search must stop after taking j draws, $j = 1, 2, \ldots, n_q$, where $n_q = [q/\alpha]$. We express the expected value of θk^γ under any strategy as follows:

$$\text{(A.17)} \qquad E\{\tilde{\theta}\cdot\tilde{k}^\gamma | \theta^*\} = \sum_{j=1}^{n_q} \left\{ (q - j\alpha)^\gamma \cdot \left(\prod_{m=1}^{j-1} H[\theta^*(q - m\alpha)] \right) \right.$$

$$\left. (1 - H[\theta^*(q - j\alpha)]) \cdot E\{\tilde{\theta} | \tilde{\theta} \geq \theta^*(q - j\alpha)\} \right\}.$$

In addition, when $H(\theta) = 1 - \theta^{-\lambda}$, then for any k,

$$\text{(A.18)} \qquad (1 - H[\theta^*(k)]) \cdot E\{\tilde{\theta} | \tilde{\theta} \geq \theta^*(k)\} = \frac{\lambda}{\lambda - 1}(\theta^*(k))^{-\lambda}.$$

Since by hypothesis $H[\theta_2^*(k)] \geq H[\theta_1^*(k)]$ for any k, the result follows from (A.17) and (A.18) under the assumption $\theta_2^*(k) < (1+x)\theta_1^*(k)$. □

The expression whose expected value we are interested in is θk^γ, which is proportional to the output produced by a firm that uses production capital k, technology θ, and labor hired according to its marginal product given these other factors at the going equilibrium wage rate. Let $\theta_y^*(\cdot)$ be the threshold function associated with the objective of maximizing expected output. Then, by definition, the expected value of θk^γ under θ_l^* cannot exceed the expected value of the same function under θ_y^*. It follows from Lemma A3, that:

$$\text{(A.19)} \qquad E\{\tilde{\theta}\cdot\tilde{k}^\gamma | \theta_y^*\} \geq E\{\tilde{\theta}\cdot\tilde{k}^\gamma | \theta_l^*\} > \theta_l^*(q)\cdot q^\gamma.$$

We now apply Lemma A5 by choosing $\theta_1^*(\cdot) = \theta_y^*(\cdot)$, and $\theta_2^*(\cdot) = \theta^*(\cdot)$. The values of $\theta_y^*(\cdot)$ and $\theta^*(\cdot)$ are obtained for $k > \alpha$ from the recursive equations, (A.7) and (A.8), in Lemma A1. By Lemma A4, since the objective corresponding to output maximization, $k\theta^\gamma$, is a concave transformation of the objective for profit maximization, $k^{1/\lambda}\theta$, we have: $\theta_y^*(k) \leq \theta^*(k)$, for all $k > 0$. It can also be shown that $\theta^*(k)/\theta_y^*(k)$ is an increasing function of k, approaching the limit $1+x = [(\lambda\gamma-\gamma)/(\lambda\gamma-1)]^{1/\lambda}$, $x>0$, as $k\to\infty$. Consequently, $\theta^*(k)/\theta_y^*(k)<(1+x)$, for all k, so that Lemma A5 applies to these two strategies. Together with (A.19) we then have:

$$\text{(A.20)}\quad E\{\tilde{\theta}\cdot\tilde{k}^{\gamma}|\theta^*\}\geq(1+x)^{-\lambda}E\{\tilde{\theta}\cdot\tilde{k}^{\gamma}|\theta_y^*\}>(1+x)^{-\lambda}\cdot\theta_l^*(q)\cdot q^{\gamma}$$

$$\to\left(\frac{\lambda\gamma-1}{\lambda\gamma-\gamma}\right)\left(\frac{1}{\alpha(\lambda\gamma+1)}\right)^{1/\lambda}\cdot q^{(\gamma+1/\lambda)},\quad \text{as } q\to\infty.$$

Using (A.20) in (3.5) yields the result in Proposition 3.1. □

Section 3. *Proof of Proposition* 4.2. Suppose that the economy is in a search phase at t with a default technology θ_t, and that the search phase has lasted continuously for the last s periods, $s>1$. As θ_t is at least as good as any technology employed by any firm at $t-s-1$, all firms at $t-s-1$ had production capital levels at which θ_t would have been acceptable, so that $\theta_t\geq\theta^*(k_{t-s-1}^i)$ for all i.

Since the economy is in a search phase at t, $\theta^*(Q_t/I)>\theta_t$. Consequently, any firm which finally accepts θ_t during period t, does so only after having attempted, but failed, to uncover a better technology, and when its remaining capital runs down to the level that makes θ_t acceptable. In particular, if any firm accepts θ_t at period t, its remaining capital, $\hat{k}$, must satisfy

$$\theta^*(\hat{k}+\alpha)>\theta_t\geq\theta^*(\hat{k}).$$

That is, with remaining capital of $\hat{k}+\alpha$ units the default technology is unacceptable, but with $\hat{k}$ it is. The final production capital of each firm at t must be greater than $\hat{k}-\alpha$, because the firm would have stopped its search, and accept θ_t, before its remaining capital dropped below $\hat{k}-\alpha$. It follows, that $k_t^i>\hat{k}-\alpha\geq k_{t-s-1}^i-\alpha$, and that $\theta_t^i\geq\theta_t\geq\theta_{t-s-1}^i$, for all i. Thus, even if all firms at t end up adopting the default technology θ_t, and even when all firms at $t-1$ used that same technology, $K_t>K_{t-s-1}-I\alpha$, where K_t is the aggregate production capital at period t. The growth rate of output between periods $t-s-1$ and t corresponding to that worst case situation is, therefore, bounded below by $(1-I\alpha/K_{t-s-1})^{\gamma}-1$, which goes to zero as K_{t-s-1} goes to infinity. □

PROOF OF PROPOSITION 4.3. Let the economy be at a no-search phase at period t. As long as the no-search phase continues over consecutive periods, the beginning of period capital stock evolves according to:

$$Q_{t+1}=\beta(1-\gamma)AN^{1-\gamma}\theta_t Q_t^{\gamma}.$$

Accepting θ_t without search means that $\theta_t>\theta^*(Q_t)>\Psi(Q_t)$, where the last inequality follows from Lemma 4.1 under the assumed conditions. By definition, $\theta_t>\Psi(Q_t)$ implies that $Q_{t+1}>Q_t$. Accordingly, as long as the no-search phase continues, we get a strictly increasing sequence of beginning of period capital stocks,

$Q_t < Q_{t+1} < Q_{t+2} <, \ldots,$. But without search, this sequence converges monotonically to Q, defined by $\theta_t = \Psi(\hat{Q})$. The increasing sequence $\{Q_s\}_{s \geq t}$ must exceed any level of Q lower than $\hat{Q}$ within finitely many periods. In particular, since $\Psi(Q) < \theta^*(Q/I)$ for all $Q > \overline{Q}$, the economy must reach a level of capital at which θ_t is no longer accepted, and start a new search phase when that happens. □

PROOF OF PROPOSITION 4.4. Let (K_t, θ_t) be the production capital and best technology employed at some initial period t, in which the default technology was θ_{t-1}. Since θ_t is the best technology employed at t, it would have been acceptable to all the firms at t, so that $\theta_t \geq \Phi(K_t, \theta_{t-1}) \equiv \text{Max}\{\theta^*(K_t/I), \theta_{t-1}\}$.

By Lemma 4.1, under the parameters assumed here, there exist $\overline{Q}$ such that $\Phi(Q, \theta_{t-1}) < \Psi(Q)$ for all $Q > \overline{Q}$. Let $\bar{\theta} = \Psi(\overline{Q}) = \Phi(\overline{Q}, \theta_{t-1})$. Then either θ_t is such that $\theta_t \geq \bar{\theta}$, or else $\theta_t < \bar{\theta}$, (see Figure 1, Panel B). We now consider each of these possibilities separately.

Case 1. There are two sub cases here: (a) $\bar{\theta} \leq \theta_t < \Psi(K_t)$, and (b) $\theta_t \geq \text{Max}\{\Psi(K_t), \bar{\theta}\}$. In both sub cases, however, $\theta_t \geq \Phi(K_t, \theta_{t-1})$. Under sub case (a) θ_t is too low to reproduce savings of K_t, even if all firms had access to that technology. Consequently, Q_{t+1} is lower than K_t, and θ_t is accepted without search in $t+1$. The economy continues on this no-search phase, converging down to $\hat{K}$, which satisfies $\Psi(\hat{K}) = \theta_t$. Under case (b), the economy also moves on a no-search phase, this time converging up to $\hat{K}$.

Case 2. $\theta_t < \bar{\theta}$. Note that this case cannot exist if $\theta_{t-1} = \bar{\theta}$. If it does exist, then $K_t < \overline{Q}$, so that $\Phi(K_t, \theta_{t-1}) > \Psi(K_t)$. Hence $\theta_t > \Psi(K_t)$. The economy must resume search at or after $t+1$. For if it did not, then it grows, for a while, on a no-search phase, converging towards the capital level $\hat{K} = \Psi^{-1}(\theta_t) < \overline{Q}$. But then it must switch into a search phase, before settling on $(\hat{K}, \theta_t)$. Consequently, the economy's capital is bounded below by $\overline{Q}$ as long as search is not resumed. If search is resumed, the economy comes out of it with a combination (K, θ) such that either $K < \overline{Q}$ and $\theta < \bar{\theta}$ again, or else it belongs to Case 1 above. Either way, growth eventually stops. □

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