The fairness-efficiency tradeoff in bargaining

Shiran Rachmilevitch*

May 11, 2014

Abstract

The main expression for the fairness-efficiency tradeoff in the bargaining literature takes the form of the *constant elasticity solutions* (CES), that "range from egalitarianism to utilitarianism." I present an alternative parametrized family of solutions, the *EED family*, that expresses a fairness-efficiency tradeoff without referring to utilitarianism. There are non-trivial connections between the CES and EED families, and these are studied in detail. I axiomatize a class of bargaining solutions that includes, as special cases, the EED family, the *proportional solutions*, and the *asymmetric Kalai-Smorodinsky solutions*.

Keywords: Efficiency; Fairness; Constant elasticity solutions.

JEL Codes: C71; C78; D61; D63.

*University of Haifa, Department of Economics, Mount Carmel, Haifa, 31905, Israel. Email: shiranrach@econ.haifa.ac.il Web: http://econ.haifa.ac.il/∼shiranrach/

1 Introduction

The tradeoff between fairness and efficiency is one of the most fundamental forces in collective decision making. Can we lend it a precise expression? This question is not merely a philosophical one, but, as the discussion below suggests, is also important from a practical standpoint.

Consider a politician who needs to chose a single alternative from a given set of available options. She faces the following difficulties when making her choice.

Difficulty 1: It is not clear what objective, if any, she should seek to maximize. For example, if her goal is to maximize the preferences of her electorate base, she may run into the problem of Arrow's impossibility—namely, that these preferences are not aggregatable.

Difficulty 2: She may very well like to be able to explain (or defend) her choices, and may therefore be reluctant to promote an abstract or complicated objective (even if this objective is "right," in the sense, for example, of representing the interests of the politician's electorate base).

I argue that a clear expression of a *fairness-efficiency tradeoff* can be useful in the context of either Difficulty. In the context of Difficulty 1, it can help the politician at an introspective level, to clarify the *meaning* of her choices; in other words, it can serve as a tool helping the politician to shape her preferences. In the context of Difficulty 2, it lends a precise sense to statements such as "alternative x is less fair than y, but more efficient," statements that can be invoked to explain and defend choice behavior.

I will address the fairness-efficiency tradeoff in a simplified version of Nash's (1950) bargaining model. A *Nash bargaining problem* can be described as a feasible-set, generically denoted by S, which consists of various utility allocations, out of which a single allocation needs to be selected. A *bargaining solution* is a selection—a rule that assigns a feasible agreement to every problem. In the sequel I will be concerned with comparing different bargaining solutions in terms of their fairness and efficiency.

In the existing literature, the formalization of the fairness-efficiency tradeoff takes the form of the following *parametrization*: the family of *constant elasticity solutions*, or CES for short (Sobel (2001), Bertsimas et al. (2012), Haake and Qin (2013)), is such that its members are parametrized by a single number, $\rho \in (-\infty, 1]$: the solution corresponding to ρ, ν^ρ, is defined by

$$\nu^\rho(S) \equiv \text{argmax}_{(x,y)\in S}[x^\rho + y^\rho]^{1/\rho}.$$

The "endpoints" of this family, the ones that correspond to $\rho = 1$ and to the limit $\rho \to -\infty$, are the utilitarian and egalitarian bargaining solutions, respectively. As ρ increases, the corresponding solution, informally speaking, assigns less importance to fairness and more importance to efficiency. The limit $\rho \to 0$ is special case, and it corresponds to the *Nash solution* (Nash, 1950).

The members of the CES family are (linearly) ordered in terms of their fairness and efficiency. The fairness criterion postulated by CES is egalitarianism, which identifies, as its name suggests, "fairness" with utility-equality; this solution also implies that the well-being of the worst off individual is maximized (Rawls, 1971). Even though egalitarianism is not the only natural manifestation of "fairness" in the bargaining model, it is a transparent and very well-accepted one.[1] By contrast, the efficiency-criterion postulated by CES, utilitarianism, is problematic: whereas utilitarianism is equivalent to Pareto optimality if utility is transferable, its foundations are less obvious in the absence of such transfers.

My goal is to offer an alternative parametrized expression of the fairness-efficiency tradeoff in the bargaining model, in a way that does not suffer from this drawback. I introduce a parametrized family of bargaining solutions, the *EED family*, that formalizes the tradeoff between fairness and efficiency without referring to utilitarianism; the efficiency-criterion on which the EED family relies differs from utilitarianism, and does not require utility transfers for its justification.

[1] An alternative to it, for example, is the criterion of *equal losses* (Chun, 1988).

There are some non-trivial connections between the EED and CES families. The former is "centered" around the *Kalai-Smorodinsky solution* (Kalai and Smorodinsky, 1975) similarly to the way that the latter is "centered" around the Nash solution. For each family, this "center" is the only *scale invariant* member of that family. It is also the only member of the family that satisfies *midpoint domination* (Sobel, 1981).

Axiomatizing the EED family remains, at the present time, an open problem. I do, however, axiomatize a superclass of it; in addition to EED, this larger class of solutions contains the *proportional solutions* (Kalai, 1977) and the *asymmetric Kalai-Smorodinsky solutions* (Dubra, 2001). As a by-product of the analysis, two novel characterizations of the Kalai-Smorodinsky solution are obtained: one which is efficiency-free and one which is symmetry-free.

The rest of the paper is organized as follows. Section 2 introduces the model, discusses the basic fairness and efficiency definitions, and presents and discusses the EED family. The superclass of this family is presented and axiomatized in Section 3, which also contains the new axiomatizations of the Kalai-Smorodinsky solution. Section 4 briefly concludes.

2 Fairness and efficiency in the bargaining model

A *bargaining problem* (a problem, for short) is a convex, compact and comprehensive set $S \subset \mathbb{R}^2_+$ that contains the origin $\mathbf{0} \equiv (0,0)$ as well as some point $x > \mathbf{0}$.[2] A *solution* is a function that choses a unique point from every problem.

The definition of "fairness" enjoys a considerable consensus in the bargaining literature—*the* fair solution is the egalitarian solution (Kalai, 1977), the one that equates the players' gains from bargaining, subject to weak Pareto optimality.

"Efficiency" is more subtle, since Pareto efficiency has only moderate implications

[2]Vector inequalities are as follows: xRy iff x_iRy_i for all i,for both $R \in \{\geq, >\}$; $x \gneq y$ iff $[x \geq y$ and $x \neq y]$. Comprehensiveness of S means that if $x \in S$ then $y \in S$, for every y that satisfies $\mathbf{0} \leq y \leq x$.

in the bargaining framework—any point on the (typically infinite) Pareto frontier of a utility-set passes Pareto's test. Therefore, the Benthamian criterion of utility-sum-maximization presents itself as a natural candidate for a *decisive* efficiency criterion. However, it is not obvious what is the justification for such maximization: if 1-to-1 utility transfers among the players are possible, then this maximization is equivalent to Pareto efficiency, but in the bargaining model these transfers are not permitted, and the equivalence disappears. One may still advocate for utility-sum-maximization on the basis that the utility-sum represents the *well-being of the group*. This idea, however, involves two big assumptions: (a) the assumption of an entity—the *group* of individuals—whose status (and, philosophically speaking, even existence) is not at all obvious, and (b) the assumption that this group's "utility" equals the sum of its members' utilities.[3] It would therefore be desirable to have an efficiency criterion which, like utility-sum-maximization, typically leads to a unique recommendation, but which does not rely on debatable philosophical assumptions. Of course, such a criterion is also required to have solid economic merits.

Consider then the following criterion: look at the (typically unique) individual whose first-best payoff is maximal, and select the agreement that gives rise to this payoff; if this individual is not unique—i.e., in case that the bargainers' first-best payoffs are the same—implement the egalitarian solution. The motivation for this procedure is best illustrated in the following context: suppose that bargaining is over the division of some resource; in this case, the favored individual is *the most efficient one*, where efficiency is defined as the ability to convert the resource into utility. In order to present this idea more formally, the following notation and definitions will be useful.

The abovementioned *egalitarian solution*, denoted as E, is defined as follows:

[3] An important debate in the social choice literature revolves around accepting (a) and arguing on the validity of (b). Specifically, this debate assumes the existence of a *social welfare function* and focuses about whether it should be linear or not. See Harsanyi (1975, 1978) and Sen (1970).

given a problem S, it selects the point x that satisfies $x_1 = x_2$ and that lies in S's (weak) *Pareto frontier*—the subset $WP(S) \equiv \{x \in S : y > x \Rightarrow y \notin S\}$.[4] The *i-th dictatorial bargaining solution* is defined as follows: for the problem S, the point chosen by this solution, denoted $D^i(S)$, is the point $x \in WP(S)$ such that $x_j = 0$ for $j \neq i$. Under D^i, player i obtains his *ideal payoff*, namely $a_i(S) \equiv \max\{x_i : x \in S\}$. Let the *endogenous dictatorship solution*, ED, be defined as follows:

$$ED(S) = \begin{cases} D^i(S) & \text{if } a_i(S) > a_j(S) \\ E(S) & \text{if } a_1(S) = a_2(S). \end{cases}$$

The solution ED is the formal manifestation of the aforementioned efficiency criterion: the most efficient individual—the one who can generate greater utility—is chosen to be the "dictator," and if this individual is not unique, then egalitarianism is applied.[5]

Given $\theta \in [0, 1)$, let μ^θ be the solution that assigns to each S the point $x \in WP(S)$ that satisfies:

$$\frac{x_2}{x_1} = [\frac{a_2(S)}{a_1(S)}]^{\frac{\theta}{1-\theta}}. \tag{1}$$

Note that $\mu^0 = E$, that the limit $\theta \to 1$ corresponds to ED, and that the solution moves continuously from the former to the latter as θ increases from zero to one. Call the family $\{\mu^\theta : \theta \in [0, 1)\}$ the *EED family* (the "E" and "ED" in this name denote the "endpoints" of the family). The "midpoint" of the EED family, the solution $\mu^{\frac{1}{2}}$, is the *Kalai-Smorodinsky solution*, KS (Kalai and Smorodinsky, 1975). This is analogous to the fact that the CES family is "centered" around the Nash solution, hereafter denoted as N.

It is well known that N is the only CES solution that satisfies *scale invariance* (a solution μ is said to have this property if for every problem S and every pair of

[4] $P(S) \equiv \{x \in S : y \gneqq x \Rightarrow y \notin S\}$ is the strong Pareto frontier.

[5] In fact, the utilitarian solution can also be viewed as expressing the idea of endogenous dictatorship—under utilitarianism, the question is how to allocate the *marginal* unit, and it is precisely the most efficient individual who obtains it.

independent positive linear transformations, $l = (l_1, l_2)$, it is true that $\mu(l \circ S) = l \circ \mu(S))$. Similarly, KS, the "center" of the EED family, is the only scale invariant EED solution.

Proposition 1. *An EED solution is scale invariant if and only if it is the Kalai-Smorodinsky solution.*

Proof. It is well known that KS is scale invariant. Conversely, consider μ^θ for an arbitrary $\theta \in [0, 1)$. Let S be a triangle with $a(S) = (1, b)$ for some $b > 0$, and consider the linear transformations $l = (l_1, l_2)$, where l_1 is the identity and $l_2(a) = \lambda a$, for some $\lambda > 1$. Suppose that $\mu^\theta(S) = (x, y)$, which means that $\frac{y}{x} = b^{\frac{\theta}{1-\theta}}$. Therefore, if this solution satisfies scale invariance, the application of l to S would imply $\frac{\lambda y}{x} = (\lambda)^{\frac{\theta}{1-\theta}} b^{\frac{\theta}{1-\theta}}$, which implies $\theta = \frac{1}{2}$. $\square$

That scale invariance pins down the center of either solution family is more than a mere technicality. The reason is that scale-invariance implies that utilities are not interpersonally comparable, and the tension between fairness and efficiency—and it does not matter whether the latter takes on the expression of utilitarianism or of endogenous dictatorship—is, at its core, a tension built on interpersonal comparisons. Indeed, the strive for fairness finds support in statements such as "it is not fair that you will gain this much and I will only gain this much," whereas efficiency finds support in statements such as "you should compromise and give up a little, because it would be efficient if I got more." Under either CES or EED, the reconciliation of these opposing forces leads to a complete elimination of the thing that allows them to exist—namely, interpersonal comparisons.

In addition to scale invariance, another property which is shared by the N and KS solutions is *midpoint domination* (Sobel, 1981), which requires that each player receives at least one half of his ideal payoff. It is well known that N is the only CES solution that satisfies midpoint domination and it is easy to check that KS is the unique such EED solution. Midpoint domination has both fairness and efficiency aspects. In terms of expected utilities, it is related to the procedure known as *randomized dictatorship*: a fair coin toss determines which player will be the "dictator," and obtain his ideal payoff. This ex-post asymmetric procedure is fair in an obvious sense—it treats the players equally, irrespectively of any characteristics that the bargaining problem may have. Midpoint domination combines fairness and efficiency, since it consists of the combination of randomized dictatorship and Pareto optimality: it is the requirement that the solution point Pareto dominates whatever that can be achieved via randomized dictatorship. Under either CES or EED, this combination of fairness and efficiency pins down the "midpoint" of the fairness-efficiency spectrum.

3 Axiomatics

Ideally, one would like to have an axiomatization of the EED family. It is not obvious how to obtain an elegant such axiomatization, and this problem remains, at the present time, open. Below I introduce and axiomatize a certain generalization of EED.

The parametrization described in (1) can be generalized as follows:

$$\frac{x_2}{x_1} = \psi(\frac{a_2(S)}{a_1(S)}), \tag{2}$$

where ψ is some continuous function. For example, $\psi \equiv 1$ corresponds to the egalitarian solution and ψ being the identity corresponds to that of Kalai and Smorodinsky. With equation (2), as opposed to equation (1), it is possible to allow for asymmetry, as manifested, for example, by the *proportional solutions* (Kalai, 1977), which correspond to ψ being constant, but not necessarily identically one.[6] Another example of a non-symmetric solution is the *asymmetric Kalai-Smorodinsky solution* (Dubra,

[6]A solution μ is proportional is there is a vector $p > \mathbf{0}$ such that $\mu(S) \equiv rp$, where r is the maximal number such that this expression is in S. The egalitarian solution is the *symmetric* proportional solution, the one corresponding to $p = (1, 1)$.

2001), which corresponds to ψ being linear but not the identity (i.e, $\psi(t) = \lambda t$ for some $\lambda \in \mathbb{R}_{++} \setminus \{1\}$).[7]

Given a continuous function ψ, let σ^{ψ} denote the solution corresponding to it as described in (2). Let $\Sigma^* \equiv \{\sigma^{\psi} : \psi$ is a continuous function from $\mathbb{R}_{++}$ to itself$\}$.

We can turn to the axiomatization of Σ^*. In the statements of the axioms below, S and T are arbitrary problems and μ is an arbitrary solution.

Independence of irrelevant alternatives (IIA): If $S \subset T$ and $\mu(T) \in S$, then $\mu(S) = \mu(T)$.

IIA is due to Nash (1950). On the one hand, it expresses a sensible idea—deletion of options that were not chosen in the first place should not affect the bargaining's outcome. On the other hand, it implies extreme insensitivity to the shape of the problem. The Kalai-Smorodinsky solution, among many others, violates IIA. However, it does satisfy the following weaker variant of it, which is due to Roth (1977b).

Restricted independence of irrelevant alternatives (RIIA): If $S \subset T$, $\mu(T) \in S$ and $a(S) = a(T)$, then $\mu(S) = \mu(T)$.

The following axiom, due to Dubra (2001), is weaker than IIA but stronger than RIIA.[8]

Homogeneous ideal independence of irrelevant alternatives (HI-IIA): If $S \subset T$, $\mu(T) \in S$ and $a(S) = ra(T)$ for some $r \leq 1$, then $\mu(S) = \mu(T)$.

[7] To be precise, the solution that corresponds to λ is referred to by Dubra as D_{λ}; the *asymmetric Kalai-Smorodinsky solution* for S is defined by Dubra to be the point $x \in P(S)$ that Pareto dominates $D_{\lambda}(S)$.

[8] Dubra (2001) calls this axiom "restricted IIA." I use a different name in order to distinguish it from Roth's axiom.

The rationale behind RIIA and HI-IIA is that IIA should only be applied to pairs of similar problems. Equality of the ideal payoffs is sufficient for this similarity according to RIIA, while equality of ideal-payoffs-ratios is sufficient according to HI-IIA. Both of these axioms are special instances of a more general idea, which is due to Thomson (1981). Thomson (1981) introduced the concept of a *reference function*, g, with respect to which an independence axiom is formulated. Given such a g, the corresponding axiom requires that IIA be applied to two problems if, in addition to the requirements of IIA, these problems share the same g-value. RIIA corresponds to $g(S) = a(S)$ and HI-IIA corresponds to $g(S) = a_2(S)/a_1(S)$.

The rest of the axioms below are well-known and intuitively clear, and so, for the sake of brevity, will not be discussed.

Pareto optimality (PO): $\mu(S) \in P(S)$.

The following axiom is due to Roth (1979).

Restricted monotonicity(RM): If $S \subset T$ and $a(S) = a(T)$, then $\mu(S) \leq \mu(T)$.

The following axiom is due to Roth (1977a).

Strong individual rationality (S.IR): $\mu(S) > \mathbf{0}$.

The following axiom is due to Kalai and Smorodinsky (1975).

Individual monotonicity (IM): If $S \subset T$, $a_j(S) = a_j(T)$ and $a_i(S) \leq a_i(T)$, then $\mu_i(S) \leq \mu_i(T)$.

The following axiom, which was discussed at the end of the previous Section, is due to Sobel (1981).

Midpoint domination (MD): $\mu(S) \geq \frac{1}{2}a(S)$.

The following axioms are due to Kalai (1977).

Homogeneity (HOM): $\mu(cS) = c\mu(S)$ for all $c > 0$.

Continuity (CONT): If $\{S_n\}$ converges to S in the Hausdorff topology, then $\mu(S_n)$ converges to $\mu(S)$.

Equipped with the axioms, we can turn to the main characterization.

Theorem 1. *A solution satisfies homogeneity, strong individual rationality, restricted monotonicity, continuity, and homogeneous ideal independence of irrelevant alternatives, if and only if it belongs to* Σ^*.

Given a problem S, let $\Delta(S) \equiv \text{conv}\{\mathbf{0}, (a_1(S), 0), (0, a_2(S))\}$.

Lemma 1. *Let S be such that $P(S) = WP(S)$. Let μ be a solution that satisfies homogeneous ideal independence of irrelevant alternatives, strong individual rationality, restricted monotonicity, and homogeneity. Then $\mu(S) = \lambda\mu(\Delta(S))$, where*

$$\lambda = max\{\lambda' : \lambda'\mu(\Delta(S)) \in S\}.$$

Proof. Make these assumptions. Let $x \equiv \mu(\Delta(S))$. By S.IR, $x > \mathbf{0}$. Let λ be the maximal number such that $\lambda x \in S$. Let $T \equiv \lambda\Delta(S)$ and let $V \equiv S \cap T$. By HOM, $\mu(T) = \lambda x$ and by HI-IIA $\mu(V) = \lambda x$. By RM, $\mu(S) \geq \lambda x$; therefore, since $P(S) = WP(S)$, $\mu(S) = \lambda x$. $\square$

Proof of Theorem 1: It is easy to verify that every solution in Σ^* satisfies the axioms. Conversely, let μ be such a solution. Let $\psi\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ be defined as follows. Given a number $b > 0$, let $\psi(b) \equiv \frac{\mu_2(\text{conv}\{\mathbf{0},(1,0),(0,b)\})}{\mu_1(\text{conv}\{\mathbf{0},(1,0),(0,b)\})}$. By S.IR $\mu_1(\text{conv}\{\mathbf{0},(1,0),(0,b)\}) > 0$, so ψ is well defined. I argue that for any smooth S equation (2) holds with $x = \mu(S)$ and the just-defined ψ. Since all that matters on the RHS of (2) is the ideal-payoffs ratios, and since μ satisfies HOM, it is enough to prove the aforementioned assertion for S's with $a_1(S) = 1$. Moreover, by CONT we may restrict attention to problems S with $P(S) = WP(S)$. Let then S be such a problem and let $b = a_2(S)$. Let $x = \mu(S)$. By Lemma 1, $\frac{x_2}{x_1} = \frac{\mu_2(\text{conv}\{\mathbf{0},(1,0),(0,b)\})}{\mu_1(\text{conv}\{\mathbf{0},(1,0),(0,b)\})} = \psi(b)$. Therefore, $\mu(S) = \sigma^\psi(S)$ for every such S. By CONT, $\mu(S) = \sigma^\psi(S)$ for every S; that is, $\mu = \sigma^\psi$. Therefore $\mu \in \Sigma^*$. $\square$

The axioms listed in Theorem 1 are independent. The Nash solution satisfies all of them but RM. The *disagreement solution*—the constant solution that assigns $\mathbf{0}$ to every problem—satisfies all of them but S.IR. A general *monotonic solution* (Peters and Tijs, 1985) satisfies all of them but HOM; such a solution assigns to every problem the intersection point of its frontier with some (possibly non-linear) increasing curve. The solution that assigns to each S the point $\frac{1}{2}KS(S)$ satisfies all of them but HI-IIA. Finally, the lexicographic extension of E—the solution that picks for each S the highest point in S (according to Pareto's ordering) that dominates $E(S)$—satisfies all the axioms but CONT.

Theorem 1 is a generalization of Theorem 3 from Kalai (1977). That theorem is obtained from our Theorem 1 by the replacement of HI-IIA by IIA and the replacement of RM by IM; the result is a characterization of the proportional solutions. Thus, by weakening the IIA and IM conditions in Kalai's theorem, a broader class of solutions obtains, of which the proportional solutions are special members. This broader class of solutions can be thought of as a class of *endogenously proportional solutions*.[9]

[9]Saglam (2013) describes an alternative notion of endogenously proportional solutions.

It is easy to see that KS is the unique solution in Σ^* that satisfies PO. Moreover, since the only rule of S.IR in the above proof is to guarantee the existence of the number λ which is described in Lemma 1 (it is not well defined if $\mu(\Delta(S)) = \mathbf{0}$) and since PO also guarantees its existence, the following result follows immediately from the analysis above.

Theorem 2. *A solution satisfies homogeneity, Pareto optimality, restricted monotonicity, continuity, and homogeneous ideal independence of irrelevant alternatives, if and only if it is the Kalai-Smorodinsky solution.*

Sketch of proof: KS satisfies all the axioms. Conversely, let μ be an arbitrary solution that satisfies them. By the arguments similar to those from the proof of Theorem 1, $\mu \in \Sigma^*$. Therefore $\mu = KS$, since KS is the only element in Σ^* that satisfies PO. $\square$

Theorem 2 strengthens Theorem 2 from Dubra (2001). The latter is a characterization of KS on the basis of scale-invariance, IM, PO, HI-IIA, and CONT. The strengthening, therefore, is on two accounts: HOM is weaker than scale invariance and RM is weaker than IM. The interesting point about the two theorems—Theorem 2 from above and Theorem 2 from Dubra (2001)—is that they both offer a characterization of KS without utilizing a symmetry axiom: the combination of the other axioms already contains "enough fairness," so symmetry follows from it as an inevitable conclusion.

One final point which is worth noting concerns midpoint domination. Recall that at the end of the previous Section we noted that KS is the only EED solution that satisfies midpoint domination. It is easy to see that, in fact, it is the only solution in Σ^* that posses this property. This is leads us to the final result of this paper.

Theorem 3. *A solution satisfies homogeneity, midpoint domination, restricted monotonicity, and homogeneous ideal independence of irrelevant alternatives, if and only if it is the Kalai-Smorodinsky solution.*

Proof. Clearly KS satisfies the axioms. Conversely, let μ be a solution that satisfies them. Since MD implies S.IR, Lemma 1 holds. In particular, it follows that $\mu(S) = KS(S)$ for every S that satisfies $P(S) = WP(S)$. Let S be an arbitrary problem. By the aforementioned conclusion $\mu(\text{conv}\{\mathbf{0}, KS(S), (a_1(S), 0), (0, a_2(S))\}) = KS(S)$. Therefore, by RM, $\mu(S) = KS(S)$. □

As opposed to Theorems 1 and 2, Theorem 3 does not require a continuity axiom. More importantly, the central aspect of Theorem 3 is that it provides KS with an efficiency-free axiomatic foundation. The merit of efficiency-freeness comes from the fact that Pareto optimality, as obvious and natural as it is in general, can actually be viewed as problematic in the bargaining context. The reason is that "Pareto optimality" is defined on the basis of (the non-existence of) Pareto improvements, in a way that can be understood as implicitly making the following assumption: if an inferior agreement is on the table of negotiations as a candidate compromise, then it will not be chosen because *the players will agree* on an improvement. Assuming the players' ability to agree on Pareto improvements is not too different from assuming the existence of a bargaining solution. This problem of "collective rationality" has originally been discussed by Roth (1977a), who provided the Nash solution with an efficiency-free axiomatic foundation.[10] In a recent paper (Rachmilevitch, 2014a) I derive an analogous result for KS. In fact, this result is the same as the above Theorem 3 with one exception—in Theorem 3 midpoint domination is utilized and scale-invariance is not, while in Rachmilevitch (2014a) the opposite is true. Together, the message delivered by these two theorems parallels the one that was discussed at the end of the previous Section: within a large class of bargaining solutions—to be specific, within Σ^*—the Kalai-Smorodinsky solution is pinned down either by scale-invariance or by midpoint domination.

[10]Subsequent such axiomatizations have been derived by Lensberg and Thomson (1988), Anbarci and Sun (2011), and Rachmilevitch (2014b).

4 Conclusion

The CES and EED families are two families of solutions to the bargaining problem that offer two alternative parameterizations of fairness and efficiency. The former family is a generalization of the Nash solution and the latter is a generalization of the Kalai-Smorodinsky solution; thus, the two alternative viewpoints on the fairness-efficiency tradeoff are naturally built on the two main pillars of bargaining theory—N and KS.

It is interesting to note that all the CES solutions satisfy IIA, but none of them, except E, satisfies RM. Similarly, all the EED solutions satisfy RM, but none of them, except E, satisfies IIA. The solution E, the sole member of the intersection CES$\cap$EED, satisfies both RM and IIA, and, as follows from the abovementioned theorem of Kalai (1977), can "almost" be characterized on the basis of these two axioms.[11]

Based on its placement within the CES/EED family, the egalitarian solution E admits two philosophical interpretations. Within CES, $E = \lim_{\rho \to -\infty} \nu^{\rho}$. Each solution in the sequence $\{\nu^{\rho}\}_{\rho > -\infty}$ implies finite tradeoffs (i.e., MRS) between the players' utilities. By contrast, in EED there is no such limit, and the "MRS" is defined, from the very onset, as zero or infinity. Thus, the CES point of view, the one that sees E as the aforementioned limit, can be understood as expressing the idea that the individual utilities can, *in principle*, be traded for one another, but the price is just to high to actually execute such a trade. By contrast, according to the EED point view, such a trade, even in principle, is out of the question: individual utilities, this approach holds, are not for sale.

[11]Kalai's characterization combines IIA with IM (a slight strengthening of RM), and some other fairly weak conditions.

References

[1] Anbarci, N., and Sun, C-J., (2011), Weakest collective rationality and the Nash bargaining solution, *Social Choice and Welfare*, **37**, 425-429.

[2] Chun, Y., (1988), The equal-loss principle for bargaining problems, *Economics Letters*, **26**, 103-106.

[3] Bertsimas, D., Farias, V.F., and Trichakis, N., (2012), On the efficiency-fairness trade-off, *Management Science*, **58**, 2234-2250.

[4] de Clippel, G., (2007), An axiomatization of the Nash bargaining solution, *Social Choice and Welfare*, **29**, 201-210.

[5] Dubra, J., (2001), An asymmetric Kalai-Smorodinsky solution, *Economics Letters*, **73**, 131-136.

[6] Haake, C-J., and Qin, C-Z., (2013), A unification of solutions to the bargaining problem, working paper.

[7] Harsanyi, J.C., (1975), Nonlinear social welfare functions, *Theory and Decision*, **6**, 311-332.

[8] Harsanyi, J.C., (1978), Bayesian decision theory and utilitarian ethics, *American Economic Review*, **68**, 223-228.

[9] Kalai, E., (1977), Proportional solutions to bargaining situations: interpersonal utility comparisons, *Econometrica*, **45**, 1623-1630.

[10] Kalai, E., and Smorodinsky M., (1975), Other solutions to Nash's bargaining problem, *Econometrica*, **43**, 513-518.

[11] Lensberg, T., and Thomson, W., (1988), Characterizing the Nash bargaining solution without Pareto-optimality, *Social Choice and Welfare*, **5**, 247-259.

[12] Moulin, H., (1983), Le choix social utilitariste. Ecole Politechnique Discussion Paper.

[13] Nash, J.F., (1950), The bargaining problem, *Econometrica*, **18**, 155-162.

[14] Peters, H.J.M, and Tijs, S.H., (1985), Characterization of all individually monotonic bargaining solutions, *International Journal of Game Theory*, **14**, 219-228.

[15] Rachmilevitch, S., (2014a), Efficiency-free characterizations of the Kalai-Smorodinsky bargaining solution, *Operations Research Letters*, **42**, 246-249.

[16] Rachmilevitch, S., (2014b), Efficiency-free foundations for the Nash bargaining solution, working paper.

[17] Rawls, J., (1971), A Theory of Justice, Harvard University Press.

[18] Roth, A.E., (1977a), Individual rationality and Nash's solution to the bargaining problem, *Mathematics of Operations Research*, **2**, 64-65.

[19] Roth, A.E. (1977b), Independence of irrelevant alternatives and solutions to Nash's bargaining problem, *Journal of Economic Theory*, **16**, 247-251.

[20] Roth, A.E. (1979), An impossibility result concerning n-person bargaining games, *International Journal of Game Theory*, **8**, 129-132.

[21] Saglam, I., (2013), Endogenously proportional bargaining solutions, *Economics Bulletin*, **33**, 1521-1534.

[22] Sen, A.K., (1970), Collective Choice and Social Welfare, San-Fransisco: Holden-Day.

[23] Sobel, J., (1981), Distortion of utilities and the bargaining problem, *Econometrica*, **49**, 597-619.

[24] Sobel, J., (2001), Manipulation of preferences and relative utilitarianism, *Games and Economic Behavior*, **37**, 196-215.

[25] Thomson, W. (1981), A class of solutions to bargaining problems, *Journal of Economic Theory*, **25**, 431-441.