Efficiency-free characterizations of the Kalai-Smorodinsky bargaining solution

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Abstract

Roth (1977) axiomatized the Nash (1950) bargaining solution without Pareto optimality, replacing it by strong individual rationality in Nash's axiom list. In a subsequent work (Roth, 1979) he showed that when strong individual rationality is replaced by weak individual rationality, the only solutions that become admissible are the Nash and the disagreement solutions. In this paper I derive analogous results for the Kalai-Smorodinsky (1975) bargaining solution.

Keywords: Bargaining; Efficiency; Kalai-Smorodinsky solution.

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1 Introduction

The *Nash bargaining solution* was introduced and characterized by Nash (1950) on the basis of four axioms: *weak Pareto optimality, symmetry, independence of equivalent*

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utility representations, and *independence of irrelevant alternatives.* Roth (1977a) showed that the Pareto axiom can be dispensed with if one restricts attention to *strongly individually rational* solutions. Moreover, in Roth (1979b) it is shown that when *weak individual rationality* replaces the Pareto axiom in the above list, precisely two solutions become admissible: the Nash solution and the *disagreement solution.* The merit of dispensing with the Pareto axiom stems from the fact that the latter can naturally be interpreted as an expression of *collective rationality*, which is unappealing in the bargaining context.[1] Axiomatizations that do not involve Pareto optimality enjoy the advantage of not alluding to collective rationality, and several authors have subsequently provided the Nash solution with further such axiomatizations (Anbarci and Sun (2011), Lensberg and Thomson (1988)). However, no such axiomatization has been provided for the *Kalai-Smorodinsky solution* (Kalai and Smorodinsky, 1975). The goal of this paper is to fill in this gap. The main motivation for filling in this gap is that it is desirable to provide foundations for bargaining solutions in a way that does not refer to collective rationality. The main result of this paper applies to a wide class of bargaining problems, including non-convex ones. An additional contribution of this result is that it highlights the importance of the *ideal point* in bargaining (this will be explained in Section 4).

The formal model is described in the next Section, the results are in Section 3, and Section 4 offers a discussion; among other things, it contains a comparison to Roth's (and Nash's) results.

2 Model

Consider the following simple version of Nash's (1950) bargaining model. There is a compact set $S \subset \mathbb{R}^2_+$ of utility-pairs, from which two players need to choose a point. If they choose (i.e., *agree* on) $x \in S$ then player i receives the utility payoff

[1] For a recent discussion on collective rationality in bargaining, see Rachmilevitch (2014).

x_i, while failing to reach an agreement results in both players receiving zero utility. It is therefore assumed that $\mathbf{0} \equiv (0,0) \in S$; namely, the null allocation is a feasible alternative (one can think of $\mathbf{0}$ as representing outside options). It is assumed that $S \cap \mathbb{R}^2_{++} \neq \emptyset$; that is, the players can jointly and strictly benefit from cooperation. Let Σ be the collection of all such sets S. An element of Σ is called a (bargaining) *problem*.

The *ideal payoff* for player i in S is $a_i(S) \equiv \max\{x_i : x \in S\}$. The point $a(S) \equiv (a_1(S), a_2(S))$ is the *ideal point* (of S). Note that $a(S) > \mathbf{0}$ for every $S \in \Sigma$;[2] if $a(S) = (1,1)$, then S is called *normalized*.

A *bargaining domain* is a subset $\mathcal{D} \subset \Sigma$. Here I consider the following domain:

$$\mathcal{B} \equiv \{S \in \Sigma : [\mathbf{0}; a(S)] \cap S \text{ is convex}\}.^3$$

Note that $\mathcal{B}$ contains all convex problems and all comprehensive problems.[4]

A *bargaining solution* on a domain $\mathcal{D}$ is any function $\mu\colon \mathcal{D} \to \mathbb{R}^2_+$ that satisfies $\mu(S) \in S$ for all $S \in \mathcal{D}$. The *Kalai-Smorodinsky solution*, KS, due to Kalai and Smorodinsky (1975), is defined by $KS(S) = \lambda a(S)$ where $\lambda = \max\{\lambda' : \lambda' a(S) \in S\}$.[5] Ever since the formal derivation of this solution by Kalai and Smorodinsky, it has been the second-best known solution after the *Nash solution*, N (Nash, 1950). The latter is defined on the domain of convex problems: for a convex $S \in \Sigma$, $N(S)$ is the (unique) maximizer of the "Nash product" $x_1 \times x_2$ over $x \in S$. Another (trivial) solution that will be referred to in the sequel is the *disagreement solution*, $D(S) \equiv \mathbf{0}$.

The model described above is simpler than what is often assumed in the literature. In its richer version, a bargaining problem is defined to be a pair (S, d), where d is some point of S that specifies the disagreement payoffs. I consider the simpler version, in which the disagreement point is normalized to the origin and all feasible

[2] Vectors inequalities: uRv iff $u_i R v_i$ for all i, for both $R \in \{\geq, >\}$; $u \gneq v$ iff $[u \geq v \;\&\; u \neq v]$.

[3] Given two vectors, x and y, the segment connecting them is denoted $[x; y]$.

[4] S is comprehensive if for every $x \in S$ the set $\{y \in \mathbb{R}^2_+ : \; y \leq x\}$ is contained in S.

[5] This solution was considered as early as 1953 by Raiffa, but without an axiomatization.

agreements specify non-negative utilities, since (a) the disagreement point will not play an important role in the sequel, and (b) the implicit assumption that bargaining depends only on individually rational points—though not without loss of generality—is a rather reasonable one.

2.1 Axioms

In the statements of all the axioms below, S and T are arbitrary problems and μ is an arbitrary solution.

Independence of irrelevant alternatives (IIA): If $S \subset T$ and $\mu(T) \in S$, then $\mu(S) = \mu(T)$.

Restricted independence of irrelevant alternatives (RIIA): If $S \subset T$, $\mu(T) \in S$ and $a(S) = a(T)$, then $\mu(S) = \mu(T)$.

IIA is due to Nash (1950) and RIIA is due to Roth (1977b). The following axiom, which is weaker than IIA but stronger than RIIA, is new to the literature.

Homogeneous ideal independence of irrelevant alternatives (HI-IIA): If $S \subset T$, $\mu(T) \in S$ and $a(S) = rT(S)$ for some $r \leq 1$, then $\mu(S) = \mu(T)$.

The rationale behind RIIA and HI-IIA is that IIA should only be applied to pairs of "similar" problems. Equality of the ideal payoffs is sufficient for this similarity according to RIIA, while equality of ideal-payoffs-ratios is sufficient according to HI-IIA. These axioms are particular manifestations of the following idea, which is due to Thomson (1981). Thomson (1981) introduced the concept of a *reference function*, g, with respect to which an independence axiom is formulated. Given such a g, the corresponding axiom requires that IIA be applied to two problems if, in addition to the requirements of IIA, these problems share the same g-value. RIIA corresponds to $g(S) = a(S)$ and HI-IIA corresponds to $g(S) = a_2(S)/a_1(S)$ (IIA corresponds to any constant g).

The following axiom is due to Roth (1979a).

Restricted monotonicity (RM): If $S \subset T$ and $a(S) = a(T)$, then $\mu(S) \leq \mu(T)$.

Let F be the set of linear functions from $\mathbb{R}^2_+$ to itself and let $\pi(a, b) \equiv (b, a)$.

Symmetry (SY): $\pi S = S \Rightarrow \mu_1(S) = \mu_2(S)$.[6]

Scale invariance (SI): $f \in F \Rightarrow \mu(f \circ S) = f \circ \mu(S)$.

Non-triviality (NT): There is an S such that $\mu(S) \neq \mathbf{0}$.

3 Results

Theorem 1. *A solution on $\mathcal{B}$ satisfies homogeneous ideal independence of irrelevant alternatives, scale invariance, symmetry, and non-triviality if and only if it is the Kalai-Smorodinsky solution.*

To prove Theorem 1, I will consider a partition of $\mathcal{B}$ into two: the degenerate domain, $\mathcal{B}^{\text{deg}} \equiv \{S \in \mathcal{B} : KS(S) = \mathbf{0}\}$, and its complement, $\mathcal{B}^+ \equiv \mathcal{B} \setminus \mathcal{B}^{\text{deg}}$.

Lemma 1. *A solution on $\mathcal{B}^{deg}$ satisfies homogeneous ideal independence of irrelevant alternatives, scale invariance, and symmetry, if and only if it is the Kalai-Smorodinsky (i.e., disagreement) solution.*

[6] S that satisfies $\pi S = S$ is called *symmetric*.

Proof. Obviously KS satisfies the axioms on $\mathcal{B}^{\deg}$. Conversely, let μ be an arbitrary solution on $\mathcal{B}^{\deg}$ that satisfies them and let $S \in \mathcal{B}^{\deg}$. By SI we can assume that S is normalized. Let $T \equiv S \cup \pi S$. Note that T is symmetric, normalized, and $T \in \mathcal{B}^{\deg}$. By SY, $\mu(T) = \mathbf{0}$. By HI-IIA, $\mu(S) = \mathbf{0} = KS(S) = D(S)$. $\square$

We can now turn to the treatment of the non-degenerate part, $\mathcal{B}^{+}$. Let $\mathcal{B}^{o}$ consist of those $S \in \mathcal{B}^{+}$ that satisfy the following:

- $\star$. There exists an open set U such that $\mathbf{0} \in U$ and $(U \cap \mathbb{R}^2_+) \subset S$.

Lemma 2. *Let μ be a solution on $\mathcal{B}^{o}$ that satisfies homogeneous ideal independence of irrelevant alternatives, scale invariance, symmetry, and non-triviality. Let $S \in \mathcal{B}^{o}$ be a normalized symmetric problem. Then $\mu(S) = KS(S)$.*

Proof. Let μ and S be as above. By SY, $\mu(S) = \lambda KS(S)$ for some $\lambda \in [0, 1]$. If $\lambda = 0$, then by SI, HI-IIA, and condition $\star$, it follows that there is a non-degenerate square $R \subset S$ such that $\mu(R) = \mathbf{0}$. But then, by SI and HI-IIA, it would follow that $\mu(Q) = \mathbf{0}$ for all $Q \in \mathcal{B}^{o}$, in contradiction to NT. Therefore $\lambda > 0$. Assume that $\lambda \in (0, 1)$. Let $r \in (\lambda, 1)$ and consider $V \equiv rS$. By SI, $\mu(V) = r\mu(S)$. On the other hand, by HI-IIA $\mu(V) = \mu(S)$. Since $S \in \mathcal{B}^{+}$, $KS(S) > \mathbf{0}$ and therefore $\mu(S) > \mathbf{0}$. This implies that $r = 1$—a contradiction. Therefore, $\lambda = 1$. $\square$

Lemma 3. *Let μ be a solution on $\mathcal{B}^{o}$ that satisfies homogeneous ideal independence of irrelevant alternatives, scale invariance, symmetry, and non-triviality. Let $S \in \mathcal{B}^{o}$ be a normalized problem. Then $\mu(S) = KS(S)$.*

Proof. Make the aforementioned assumptions. Let $T \equiv S \cup (\pi S)$. Note that T is normalized and symmetric, and therefore, by Lemma 2, $\mu(T) = KS(T)$. Moreover, $KS(T) = KS(S)$;[7] therefore, by HI-IIA, $\mu(S) = KS(S)$. $\square$

[7] Note that $KS(S) = (x, x)$ and $KS(T) = (y, y)$ for some $x \leq y$. Assume by contradiction that $x < y$. Then $(y, y) \notin S$. But then $(y, y) \in \pi S$, and since $\pi(\pi S) = S$, we get $(y, y) \in S$—a contradiction.

Lemma 4. *Let μ be a solution on $\mathcal{B}^o$ that satisfies homogeneous ideal independence of irrelevant alternatives, scale invariance, symmetry, and non-triviality. Let $S \in \mathcal{B}^o$. Then $\mu(S) = KS(S)$.*

Proof. Follows from SI and Lemma 3. $\square$

Equipped with the lemmas, we can turn to Theorem 1's proof.

Proof of Theorem 1: Clearly KS satisfies the axioms. Conversely, let μ be a solution that satisfies them and let $S \in \mathcal{B}$. If $S \in \mathcal{B}^{\text{deg}}$ then $\mu(S) = KS(S)$ follows from Lemma 1. Suppose then that $S \in \mathcal{B}^+$. If S satisfies condition $\star$, then by Lemma 4 $\mu(S) = KS(S)$. If this condition is not satisfied by S, then it is satisfied by $S' = S \cup (U \cap \mathbb{R}^2_+)$, where U is an arbitrary small open set centered around the origin. By Lemma 4 $\mu(S') = KS(S') = KS(S)$. Hence, by HI-IIA, $\mu(S) = KS(S)$. $\square$

The only place in the proof where the domain restriction to $\mathcal{B}$ plays a role is in the part of Lemma 2's proof where the possibility $\lambda \in (0,1)$ is ruled out. Without this domain restriction, the number $r \in (\lambda, 1)$ may not exist. To see this, look, for example, at the problem $S^* = \{(x,y) : \mathbf{0} \leq (x,y) \leq (2,2), y \leq \frac{1}{x}\} \cup \{(2,2)\}$. Suppose that $\mu(S^*) = (1,1) = \frac{1}{2}KS(S^*)$; namely, $\lambda = \frac{1}{2}$. In this case, if $rKS(S^*) \in S^*$ for some $r > \frac{1}{2}$, then $r = 1$.

The domain restriction to $\mathcal{B}$ is nevertheless important. If one allows for cases where $[\mathbf{0}; a(S)] \cap S$ is not convex, and so the segment $[\mathbf{0}; KS(S)]$ can be partitioned into several closed components, the solution that picks the highest point from the first (i.e., lowest) component satisfies all of Theorem 1's axioms but is different from KS. Moreover, Theorem 1 does not hold on the domain of convex problems. Indeed, the Nash solution, which is well-defined on this domain, satisfies all of Theorem 1's axioms.[8] However, with the addition of RM, a similar result obtains also for this

[8]The step where the richness of $\mathcal{B}$ is crucial for Theorem 1's proof is in Lemma 3: typically, a set

domain.

Theorem 2. *A solution on the domain of convex problems satisfies restricted monotonicity, homogeneous ideal independence of irrelevant alternatives, scale invariance, symmetry, and non-triviality if and only if it is the Kalai-Smorodinsky solution.*

Theorem 2's proof is very similar to that of Theorem 1, and it is therefore omitted. The axioms in Theorems 1 and 2 are independent. The solution $\mu(S) = \frac{1}{2}KS(S)$ satisfies all of them but HI-IIA. Kalai's (1977) *egalitarian solution*, E, which picks for every S the maximal point in the intersection $S \cap \{(r, r) : r \geq 0\}$, satisfies all of them but SI. The *i-th dictatorial solution*—the maximal element in S of the form re^i, where $e^i_i = 1$ and $e^i_j = 0$—satisfies all of them but SY. The Nash solution satisfies all the axioms from Theorem 2 but RM. Finally, the disagreement solution satisfies all of them but NT; in fact, as the following corollary states, this is the only such solution.

Corollary 1. *A solution on $\mathcal{B}$, μ, satisfies homogeneous ideal independence of irrelevant alternatives, scale invariance, and symmetry if and only if $\mu \in \{KS, D\}$.*

Proof. Obviously D and KS satisfy the axioms. Conversely, let μ be a solution that satisfies them. If μ satisfies NT, then, by Theorem 1, $\mu = KS$. Otherwise, $\mu = D$. $\square$

Similarly we have,

Corollary 2. *A solution on the domain of convex problems, μ, satisfies restricted monotonicity, homogeneous ideal independence of irrelevant alternatives, scale invariance, and symmetry if and only if $\mu \in \{KS, D\}$.*

like $T = S \cup (\pi S)$, which is essential for this lemma's proof, is not convex. Note that $T = S \cup (\pi S)$ is also invoked in the proof of Lemma 1; however, the importance of this fact in the discussion about the bargaining domain is relatively less significant, since Lemma 1 only applies to the degenerate domain.

4 Discussion

The immediate point of reference to the results of this paper is the results of Roth (1977a, 1979b), that show that in Nash's original axiom list the Pareto axiom can be replaced by either (*i*) strong individual rationality or (*ii*) the combination of weak individual rationality and "not disagreement." Theorem 2 above offers a contribution which is analogous to (*ii*) in regards to Kalai and Smorodinsky's axiom list. NT is a formalization of the requirement "not disagreement." HI-IIA is the counterpart of weak individual rationality; of course, it is a "counterpart" in a rather formal and minimal sense, as HI-IIA and weak individual rationality are neither logically nor conceptually related; however, each plays the role of an axiom which is logically independent of the Pareto axiom, but that can replace it in the original characterization (of Nash/Kalai-Smorodinsky) when one adds the "not disagreement"/NT requirement.

There is no logical relation between Theorem 2 and the original result of Kalai and Smorodinsky, and there is no such relation between Roth's results and Nash's theorem. HI-IIA and NT are not logically comparable to the Pareto axiom, hence Theorem 2 is neither stronger nor weaker than the result of Kalai and Smorodinsky.[9] The Pareto axiom is also not logically comparable to (weak or strong) individual rationality, hence none of the results of Roth is logically comparable to that of Nash.[10]

Both Theorem 1 and Theorem 2 highlight the importance of the ideal point in

[9]Kalai and Smorodinsky make use of the following axioms: the Pareto axiom, SY, SI, and RM (in the statement of their theorem they do not list RM but a related axiom—*individual monotonicity*—which is actually stronger than RM; however, RM suffices for the proof of their theorem).

[10]For example, a *utilitarian solution*—any solution that picks a maximizer of the players' sum of utilities—satisfies the Pareto axiom but fails to satisfy even weak individual rationality (this example, of course, is meaningful only in the context of the richer version of the model, in which the disagreement point is not fixed at the origin, and non-individually rational alternatives exist). The solution that picks the average point between the Nash solution point and the disagreement point satisfies strong individual rationality, but is clearly not Paretian.

bargaining and its close connection to the Kalai-Smorodinsky solution. This idea manifests itself in the form of HI-IIA, an axiom that refers to the ideal point explicitly, and sets a restriction whose geometrical content is very close to that of the Kalai-Smorodinsky solution. In purely logical terms, this axiom can be viewed as a weakening of Nash's IIA and a strengthening of Roth's RIIA. More substantially, it is an independence axiom which is formulated on the basis of a reference function, *à la* Thomson (1981).

Emphasizing the role of the ideal point in the way that HI-IIA does, namely through a generalization of a known axiom, can be accomplished in an alternative similar way. Consider the analogous generalization of RM: say that a solution μ satisfies *homogeneous idea monotonicity* (HIM) if $\mu(S) \leq \mu(T)$ whenever $S \subset T$ and $a(S) = ra(T)$ for some $r \leq 1$. It is easy to verify that the Kalai-Smorodinsky solution is the only one that satisfies HIM and the following mild requirement: whenever the ideal point is feasible, it is selected by the solution.[11]

Finally, it should be noted that an important aspect of Theorem 1's contribution is that it offers an efficiency-free result for a large domain, without imposing convexity on the bargaining problem.[12]

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[11] This result holds both on the grad domain $\mathcal{B}$ as well as on the domain of convex problems. Its proof is immediate and is therefore omitted.

[12] Non-convexities in bargaining can arise due to a variety of reasons, and they have been discussed extensively in the literature. See Conley and Wilkie (1991,1994,1996), Denicolò and Mariotti (2000), Hougaard and Tvede (2003, 2010), Mariotti (1998), Peters and Vermeulen (2012), Qin et al. (2012), Xu and Yoshihara (2006, 2013), and Zhou (1997).

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