Games with countably many players

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Abstract

I study games in which there are countably many players, each of whom has finitely many pure strategies. Special attention is given to symmetric such games: I derive alternative sufficient conditions for the existence of a symmetric Nash equilibrium in symmetric games, I study the relation between approximate and exact equilibria in symmetric games, and I construct an example of a symmetric game in which all Nash equilibria are asymmetric.

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1 Introduction

In this paper I consider games in which there are countably many players, each of whom has finitely many pure strategies. The only difference relatively to Nash's (1951) classic model, therefore, is the assumption of a countable infinity of players.[1]

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[1] An alternative way to model large-scale interactions is to assume a continuum of players (see the review in Khan and Sun (2002)). As opposed to models with a continuum of players, the approach

Our point of departure is the following game, which is due to Peleg (1969). The set of players is $\mathbb{N}$, each player has the set of pure strategies $\{0, 1\}$, and each player i's preferences over pure profiles $a \in \{0, 1\}^{\mathbb{N}}$ are given by the following utility function:

$$u_i(a) = \begin{cases} a_i & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\ -a_i & \text{otherwise} \end{cases}$$

The sum $\sum_{j=1}^{\infty} a_j$ is finite if and only if there is a finite number of 1's. Since each a_j is an independent random variable, the occurrence of the event $\{\sum_{j=1}^{\infty} a_j < \infty\}$ depends on a countable sequence of independent random variables; since it is invariant to the realization of any finite number of them, it is a *tail event.* Kolmogorov's 0-1 Law (henceforth, the 0-1 Law) states that the probability of a tail event is either zero or one.[2] It follows from the 0-1 Law that this game does not have a Nash equilibrium. To see this, let p denote the probability of $\{\sum_{j=1}^{\infty} a_j < \infty\}$ in a putative equilibrium. If $p = 1$ then the unique best-response of each player is to play 1, which implies $p = 0$. If, on the other hand, $p = 0$, then each player's unique best-response is to play 0, which implies $p = 1$.

Utility discontinuity in the above game is inevitable, as Peleg (1969) proved that continuity of the utility functions guarantees equilibrium existence. Besides discontinuity, the utility functions in Peleg's game have an additional important feature, hereafter called *co-finiteness*: given his own-action a_i and given two alternative profiles of the actions of the others, say a_{-i} and a'_{-i}, players i's utility is the same (namely $u_i(a_i, a_{-i}) = u_i(a_i, a'_{-i})$), provided that a_{-i} and a'_{-i} differ in a finite number of coordinates. Finally, equilibrium non-existence in Peleg's game is strong, in the following sense: not only a Nash equilibrium fails to exist, even an ϵ equilibrium (Radner (1980)) does not exist, for all sufficiently small $\epsilon > 0$.

taken in the current paper offers a look at large games, which, conceptually speaking, is closer to the finite-game benchmark. For more on the conceptual and technical merits of the countable-player-set framework, see Voorneveld (2010).

[2]See Billingsley (1995).

Our first result is that the co-finiteness and continuity of a utility function, while being logically unrelated (neither implies the other) are exclusive, in the following sense: a utility function is *trivial*—namely, independent of the actions of the others—if and only if it is both co-finite and continuous. This is the content of Theorem 1. The utility discontinuity in Peleg's game is therefore a logical necessity, since this utility is co-finite, but certainly not trivial.

The remainder of the results concerns symmetric games. It is shown (in Theorem 2) that if a symmetric co-finite game[3] has an ϵ equilibrium for all $\epsilon > 0$, then it also has a Nash equilibrium. Non-existence of approximate equilibria in Peleg's game is therefore inevitable, as this game is symmetric and co-finite, but does not poses a Nash equilibrium. It is also shown (in Theorem 3) that if a symmetric co-finite game has a Nash equilibrium, then it also has a symmetric one, and (in Theorem 4) that if a symmetric game has a continuous utility function then it has a symmetric Nash equilibrium.

Besides these Theorems, the paper contains an example of a symmetric game in which Nash equilibria exist, but all of them are asymmetric. This example complements the ones of Fey (2012), who constructed games with finitely many players and infinitely many pure strategies, in which Nash equilibria exist, but all of them are asymmetric. The game in our example is not co-finite and not continuous. As follows from the above Theorems, these facts are inevitable: had the game been continuous, then, in view of Theorem 4, it would have possessed a symmetric Nash equilibrium; since the game does have (an asymmetric) equilibrium, co-finiteness would also have implied the existence of a symmetric equilibrium, due to Theorem 3.

The paper is organized as follows. Section 2 describes the model, Section 3 contains a basic result about utility functions, Section 4 discusses symmetric games, and Section 5 is dedicated to the aforementioned example.

[3] By a symmetric co-finite game it is meant a symmetric game whose common utility function is co-finite. Precise definitions will be given in the next Section.

2 Model

A game in normal-form is a tuple $G = [N, (A_i)_{i \in N}, (u_i)_{i \in N}]$, where N is the set of players, A_i is the set of player i's pure strategies (or actions), and $u_i \colon \Pi_{i \in N} A_i \to \mathbb{R}$ is i's utility function, defined on pure action profiles. A mixed strategy for i, generically denoted by α_i, is a probability distribution over A_i. In the sequel, I will use the word *game* to denote such a tuple G such that (1) N is countably infinite, and (2) A_i is finite for each $i \in N$.

A profile of mixed strategies is denoted by α and player i's expected utility under α is denoted by $U_i(\alpha)$. A *Nash equilibrium* is a profile α such that the following holds for each i: $U_i(\alpha) \geq U_i(\alpha')$, where α' is any alternative profile that satisfies $\alpha'_j = \alpha_j$ for all $j \in N \setminus \{i\}$. A Nash equilibrium α is *pure* if for each i there is an $a_i \in A_i$ such that $\alpha_i(a_i) = 1$.

A game is *symmetric* if there is a set A such that $A_i = A$ for all $i \in N$ and the preferences of each player i over elements of A^N are given by a two-argument function, $u(x, a)$, that satisfies the following: its first argument is i's own-action, its second argument is the profile describing the actions of players $j \neq i$, and, additionally, $u(x, a) = u(x, b)$ for any own-action x and any a and b for which there is a permutation on $N \setminus \{i\}$, π, such that $b_j = a_{\pi(j)}$ for all $j \neq i$. That is, each player i cares about what actions his opponents are playing, but not about who is playing what action. A Nash equilibrium α of a symmetric game is *symmetric* if $\alpha_i = \alpha_j$ for all $i, j \in N$.

A utility function u_i is *co-finite* if for any own-action x and any two action profiles of the other players, a and b, the following is true: if $|\{j : a_j \neq b_j\}| < \infty$ then $u_i(x, a) = u_i(x, b)$; the utility u_i is *continuous* if it is continuous in the product topology; it is *trivial* if it is independent of a_{-i}. A game is *co-finite* (resp. *continuous*, *trivial*) if each of its utility functions u_i has the associated property.

3 A basic result

Co-finiteness and continuity are key in the analysis carried out in the sequel. They are "truly alternative" to one another, in the sense that if the utility functions in a game are both co-finite and continuous, then this game is merely a collection of independent decision problems.

Theorem 1. *A utility function is trivial if and only if it is continuous and co-finite.*[4]

Proof. Wlog, suppose that the player set is $N = \mathbb{N}$ and look at player $i = 1$. Let $u \equiv u_1$. Suppose first that u is trivial. It is obviously co-finite. Also, since player 1 (in fact, every player) has finitely many pure strategies, it follows that u is continuous. Conversely, suppose that u is continuous and co-finite, and assume by contradiction that $|u(x,a) - u(x,b)| \equiv \Delta > 0$ for some x, a, and b. By continuity there is a $N^* \in \mathbb{N}$ such that the following is true for every profile c: if $c_j = a_j$ for all $j < N^*$ then $|u(x,a) - u(x,c)| < \Delta$. In particular, we can apply this observation to the following profile c:

$$c_j = \begin{cases} a_j & \text{if } j < N^* \\ b_j & \text{if } j \geq N^* \end{cases}$$

By co-finiteness $|u(x,a) - u(x,c)| = |u(x,a) - u(x,b)| < \Delta$, a contradiction. □

Note that continuity and co-finiteness are logically unrelated. To see that co-finiteness does not imply continuity, consider the following game, in which the set of players is $\mathbb{N}$ and each pure-strategies set is $\{0,1\}$. In this game, a player's utility from the profile a is one if a contains infinitely many 0's and infinitely many 1's, and otherwise his utility is zero. This is a (symmetric) game with a discontinuous co-finite utility. To see that continuity does not imply co-finiteness, consider the game in which the player set is $\mathbb{N}$, pure strategies are $\{0,1\}$, and the utility from the profile $a \in \{0,1\}^{\mathbb{N}}$ is $\frac{1}{1+|\{i:a_i=1\}|}$.

[4] I owe this result to Igal Milchtaich.

4 Symmetric games

Theorem 2. *Let G be a symmetric co-finite game. If G has an ϵ equilibrium for all $\epsilon > 0$, then it also has a Nash equilibrium. Moreover, this Nash equilibrium is pure.*

Proof. Let G be a game as above. Let A be its (finite) set of pure strategies. Given $\epsilon > 0$, let $\alpha = \alpha(\epsilon)$ be an ϵ equilibrium. Let $\Pr_\alpha$ denote the probability measure that α induces on A^N, where N is the set of players. Given a non-empty $S \subset A$, let $E(S)$ be the event "each element of S is realized infinitely many times." Since $E(S)$ is a tail event, its $\Pr_\alpha$-measure is either zero or one.[5] That is, $\emptyset \neq S \subset A \Rightarrow \Pr_\alpha(E(S)) \in \{0, 1\}$.

Claim: There is an $S \subset A$ such that $\Pr_\alpha(E(S)) = 1$.

Proof of the Claim: Let $\{S_1, \cdots, S_K\}$ be the non-empty subsets of A ($K = 2^{|A|} - 1$). Since $A^N = \cup_{k=1}^K E(S_k)$, the falseness of the Claim implies $1 \leq \sum_{\cup_{k=1}^K E(S_k)} \Pr_\alpha(E(S)) = 0$, a contradiction.

Let $X \equiv \cup_{\{\Pr_\alpha(E(S))=1\}} S$. By the Claim, $X \neq \emptyset$. The set X consists of all pure actions that occur infinitely many times in the α-equilibrium with probability one. Suppose that $X = \{x_1, \cdots, x_L\}$. Let a be the following profile:

$$a = (x_1, \cdots, x_L, x_1, \cdots, x_L, \cdots, x_1, \cdots, x_L, \cdots).$$

Look at a particular player i. With probability one the behavior of the others is given by a profile, b, that satisfies one of the following: (1) b is obtained from a by a permutation, or (2) there is a finite set of coordinates, J, such that the sub-profile $(b_j)_{j \notin J}$ is obtained from a by a permutation.[6] By symmetry and co-finiteness, every

[5] Peleg's game is built with reference to the set $S = \{1\}$.

[6] Note that (1) is a particular manifestation of (2)—the one corresponding to $J = \emptyset$.

x in the support of i's strategy is an ϵ-maximizer of $u(.,a)$.[7] Since this is true for every player i, it follows that every $x \in X$ is an ϵ-maximizer of $u(.,a)$. It therefore follows that a is a pure ϵ-equilibrium.

Both X and a depend on ϵ: $X = X(\epsilon)$ and $a = a(\epsilon)$. Since A is finite there is a sequence $\{\epsilon\} \downarrow 0$ such that $a(\epsilon) = a^*$ for all ϵ in the sequence. It is easy to see that a^* is a pure Nash equilibrium. $\square$

Co-finiteness is important in Theorem 2: the fact that a symmetric game has an ϵ equilibrium for all $\epsilon > 0$ does not, by itself, imply that it also has a Nash equilibrium. The following game exemplifies this.

Let G^* be the following game. The player and action sets are $\mathbb{N}$ and $\{0, 1\}$, respectively, and the utility function is:

$$u_i(a) = \begin{cases} \frac{a_i}{1+|\{k: a_k = 1\}|} & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\ -a_i & \text{otherwise} \end{cases}$$

Note that this utility function is obtained from that of Peleg's game by a relatively minor change: replacing a_i by $\frac{a_i}{1+|\{k: a_k = 1\}|}$. That this game does not have a Nash equilibrium follows from precisely the same arguments as the ones from Peleg's game. Nevertheless, G^* has an ϵ equilibrium for all $\epsilon > 0$.

Proposition 1. *The game G^* has an ϵ equilibrium for all $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. Let m be such that $\frac{1}{1+m} < \epsilon$. Consider the following (pure) strategy profile: each player in $\{1, \cdots, m\}$ plays the action 1, and every other player plays the action 0. Obviously, each player $i \leq m$ is playing a best-response; each $i > m$ can only improve his payoff by $\frac{1}{2+m} < \epsilon$ via a unilateral deviation. $\square$

Symmetry is also important in Theorem 2: the fact that a co-finite game has an ϵ equilibrium for all $\epsilon > 0$ does not, by itself, imply that it also has a Nash equilibrium. The following game exemplifies this.

[7] That is, $u(x,a) \geq -\epsilon + \sup_y u(y,a)$.

Let G^{**} be the following game. The player and action sets are $\mathbb{N}$ and $\{0,1\}$, respectively, and the utility functions are the following:

$$u_1(a) = \begin{cases} a_1 & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\ -a_1 & \text{otherwise} \end{cases}$$

The utility for any other player n is as follows:

$$u_n(a) = \begin{cases} a_n & \text{if } a_l = 1 \text{ for all } l < n \\ -a_n & \text{otherwise} \end{cases}$$

Proposition 2. *G^{**} does not have a Nash equilibrium.*

Proof. Assume by contradiction that $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ is a Nash equilibrium. Look at player 1. If he plays the pure action 1 (namely, if $\alpha_1(1) = 1$), then player 2 necessarily plays his unique best response—the pure action 1; subsequently, it is easy to see that every player n plays the action 1 with certainty. But in this case player 1 is not playing a best response, in contradiction to equilibrium. If player 1 plays the pure action 0 then player 2 necessarily plays the action 0 as well, with certainty; subsequently, it is easy to see that every player n plays the action 0 with certainty. But in this case player 1 is not playing a best response, in contradiction to equilibrium. Therefore player 1 strictly mixes.

Let I be the set of players $i > 1$ who do not play the action 1 with certainty; that is, $I \equiv \{i \in \mathbb{N} : i > 1, \alpha_i(1) < 1\}$. Obviously $I \neq \emptyset$; otherwise, player 1 would not mix, but play the pure action 0. Let $i^* \equiv \min I$. Since $\alpha_{i^*}(1) < 1$, it follows that $\alpha_1(1) \leq \frac{1}{2}$ ($\alpha_1(1) > \frac{1}{2}$ implies that i^*'s unique best-response is the action 1).

Case 1: $\alpha_1(1) < \frac{1}{2}$. Here i^*'s unique best-response is the action 0. This implies that each $i > i^*$ also plays the action 0 with certainty. Therefore player 1 is not playing a best-response: he should switch to the pure action 1.

Case 2: $\alpha_1(1) = \frac{1}{2}$. Since $\alpha_{i^*}(1) < 1$, it follows that the unique best-response of player $j = i^* + 1$ is to play the pure action 0. This implies that each $i > i^*$ also plays the action 0 with certainty. Therefore player 1 is not playing a best-response: he should switch to the pure action 1.[8] □

Despite not having a Nash equilibrium, G^{**} has an ϵ equilibrium, for any $\epsilon > 0$.

Proposition 3. *G^{**} has an ϵ equilibrium, for any $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. The following is an ϵ equilibrium. Let player 1 play the pure action 1. Let $N > 2$ be an integer, to be defined precisely shortly. Let each player $i \in \{2, \cdots, N\}$ play the action 1 is probability $1 - \epsilon$ and play 0 otherwise. Given that player 1 plays 1, the best-response of player 2 is also to play 1, so the aforementioned strategy is an ϵ best-response for him. The same is true for players 3, 4, and so on—as long at the probability that all $l < i$ play the action 1 is at least 0.5, player i's best-response is to play the pure action 1, and therefore the aforementioned strategy is an ϵ-best response. At some i—call it N—the probability that all $l < i$ play the action 1 is less than 0.5; from this N onwards, let each player play his best-response, namely 0. Note that player 1 is also playing his best-response, hence the above profile is an ϵ equilibrium. □

Since Theorem 2 considers equilibria of symmetric games, it is natural to ask whether the pure equilibrium whose existence it guarantees is also a symmetric equilibrium. As the following example shows, the answer is negative: the fact that a symmetric co-finite game has a pure equilibrium does not imply that it has a pure symmetric equilibrium.

Let G^{***} be the following game. The player and action sets are $\mathbb{N}$ and $\{0, 1\}$, respectively, and utility from pure profiles is as follows:

- If there are infinitely many 0's and infinitely many 1's, then a player's utility is one.
- If everybody play the same action, then a player's utility is zero.

[8] Note that this proof is direct, in the sense that it does not rely on Kolmogorov's 0-1 Law.

- If there are exactly k appearances of some action $x \in \{0,1\}$, where $0 < k < \infty$, then the utility of a player whose action is x is one, and otherwise it is zero.

G^{***} is co-finite and it has infinitely many non-symmetric pure Nash equilibria—every profile with infinitely many occurrences of each action is an equilibrium. Obviously, it does not have a pure symmetric equilibrium. It does, however, have a non-pure symmetric equilibrium: if each player plays each action with equal probability, a symmetric equilibrium obtains, because with probability one the realized profile has infinitely many occurrences of either action. Existence of this symmetric equilibrium is a consequence of the following general result.

Theorem 3. *If a symmetric co-finite game has a Nash equilibrium, then it also has a symmetric Nash equilibrium.*

The proof of Theorem 3 is very close to that of Theorem 2, and is therefore deferred to the Appendix. Finally, the following result provides a sufficient condition for the existence of a symmetric Nash equilibrium; it parallels that of Moulin (1986), who proved essentially the same result for finite games.

Theorem 4. *Every continuous symmetric game has a symmetric Nash equilibrium.*

Proof. Let A be the common (finite) set of pure strategies and, wlog, let $\mathbb{N}$ be the player set. Define $BR_i(\xi)$ to be the set of i's best-responses against the profile where all $j \neq i$ plays the strategy $\xi \in \Delta(A)$. By continuity $BR_i(\xi)$ is closed for all $\xi \in \Delta(A)$, and it is also non-empty and convex. By symmetry, $BR_i = BR_j \equiv BR$ for all i, j. Therefore, $BR\colon \Delta(A) \rightrightarrows \Delta(A)$ satisfies the conditions of Kakutani's fixed point theorem, and so there is an ξ^* such that $\xi^* \in BR(\xi^*)$. Clearly, the profile α, where $\alpha_i = \xi^*$ for all i, is a symmetric Nash equilibrium. $\square$

5 A symmetric game with only asymmetric equilibria

Let $\tilde{G}$ be the following game. The player and action sets are $\mathbb{N}$ and $\{0,1\}$, respectively, and the utility of player i from the pure profile a is as follows:

$$u_i(a) = \begin{cases} -1 & \text{if } a_j = 1 \text{ for all } j \in \mathbb{N} \\ a_i & \text{otherwise} \end{cases}$$

Proposition 4. *$\tilde{G}$ has a Nash equilibrium.*

Proof. Let $p \in (0,1)$ and let n be an integer such that $p^{n-1} = 0.5$. Consider the following strategy: all players in $\{1, \cdots, n\}$ play the action 1 with probability p and play 0 otherwise, and all other players play the pure action 1. I argue that each i is playing a best-response; namely, that this is a Nash equilibrium. To see this, let us look separately at a typical player $i \leq n$ and at a typical player $i > n$.

$i \leq n$. From the standpoint of this player i, the probability of the event that all players in $\mathbb{N} \setminus \{i\}$ take the action 1 is $p^{n-1} = 0.5$. Therefore, if he plays the action 1 his corresponding expected utility is $0.5(-1) + 0.5(1) = 0$, while playing 0 gives the sure payoff zero. Therefore mixing (according to p) is a best-repose.

$i > n$. From the standpoint of this player i, the probability of the event that all players in $\mathbb{N} \setminus \{i\}$ take the action 1 is $p^n = 0.5p$. Therefore, if he plays the action 1 his corresponding expected utility is $0.5p(-1) + (1 - 0.5p)(1) = 1 - p$. The action 0 gives the payoff zero, hence 1 is indeed a best-response. □

Despite having Nash equilibria and being a symmetric game, $\tilde{G}$ does not have a symmetric equilibrium. This fact parallels the results of Fey (2012), who constructed symmetric games with infinitely many *strategies* and finitely many players, in which equilibrium exists, but all equilibria are asymmetric.

Proposition 5. *$\tilde{G}$ does not have a symmetric Nash equilibrium.*

Proof. Assume by contradiction that there is a symmetric Nash equilibrium. Let q be the probability that a given player is playing the action 1 under this equilibrium. Clearly, $q \notin \{0, 1\}$. This means that each player is indifferent between the two actions. Consider an arbitrary player, i. Let p be the probability that $a_j = 1$ for all $j \neq i$. The indifference conditions for player i is $p(-1) + (1 - p) = 0$, and so $p = 0.5$. However, $p = \lim_{n\to\infty} q^n = 0$, a contradiction. □

It is easy to see that $\tilde{G}$ is not co-finite. For example, if all $i \geq 3$ take the action 1, we cannot deduce player 1's payoff from the knowledge of his own action; for instance, if he also plays the action 1 in the aforementioned situation then his payoff would either be -1 or $+1$, depending on player 2's action. Also, note that the fact that $\tilde{G}$ is not co-finite follows from the combination of Theorem 3 and Propositions 4 and 5: if $\tilde{G}$ were co-finite, then, since it has Nash equilibrium (as proved in Proposition 4) it would also poses a symmetric Nash equilibrium (as Theorem 3 dictates), in contradiction to Proposition 5. That $\tilde{G}$ is not continuous is also easy to see. That discontinuity is inevitable also follows from the combination of Theorem 4 and Proposition 5.

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Appendix

Proof of Theorem 4: Let G be a game as above, A be its strategy set, and α be a Nash equilibrium of it. $\Pr_\alpha$ denotes the probability measure that α induces on A^N, where N is the set of players. Given a non-empty $S \subset A$, let $E(S)$ be the event "each element of S is realized infinitely many times." By the 0-1 Law: $\emptyset \neq S \subset A \Rightarrow \Pr_\alpha(E(S)) \in \{0, 1\}$. It follows from the proof of Theorem 2 that there is an $S \subset A$ such that $\Pr_\alpha(E(S)) = 1$. Let $X \equiv \cup_{\{\Pr_\alpha(E(S))=1\}} S$. By the Claim, $X \neq \emptyset$. The set

X consists of all pure actions that occur infinitely many times in the α-equilibrium with probability one. Suppose that $X = \{x_1, \cdots, x_L\}$. Let a be the following profile: $a = (x_1, \cdots, x_L, x_1, \cdots, x_L, \cdots, x_1, \cdots, x_L, \cdots)$. Look at a particular player i. With probability one the behavior of the others is given by a profile, b, that satisfies one of the following: (1) b is obtained from a by a permutation, or (2) there is a finite set of coordinates, J, such that the sub-profile $(b_j)_{j \notin J}$ is obtained from a by a permutation. By symmetry and co-finiteness, every x in the support of i's strategy is a maximizer of $u(., a)$. Since this is true for every player i, it follows that every $x \in X$ is an a maximizer of $u(., a)$. It therefore follows that the strategy that randomizes uniformly on X is a symmetric equilibrium strategy. $\square$

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