Kalai-Smorodinsky-Nash robustness

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Abstract

A bargaining solution satisfies *Kalai-Smorodinsky-Nash robustness* (KSNR) if it guarantees that each player receives at least the minimum of the payoffs he would have received under the Nash and Kalai-Smorodinsky solutions. The Nash solution is uniquely characterized by KSNR and IIA, and the Kalai-Smorodinsky solution—by KSNR and individual monotonicity. The axiom KSNR is related to meta bargaining (van Damme (1986)), to interpersonal utility comparisons (Kalai (1977)), and to preferences over bargaining solutions (Border and Segal (1997)).

Keywords: Bargaining; Kalai-Smorodinsky solution; Nash solution.

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1 Introduction

The bargaining problem (due to Nash (1950)) is a fundamental problem in economics.[1] The two most prominent solutions to this problem, the Nash solution (1950) and the

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[1] See Thomson (1994) for a comprehensive survey.

Kalai-Smorodinsky solution (1975), both have their merits, in terms of their axiomatic foundations as well as in terms of the non-cooperative implementations they enjoy.[2] However, they are different solutions, and so there exist bargaining problems on which they disagree. What can we do (or, perhaps, what *should* we do) in these cases? Does there exist a systematic method for solving all bargaining problems, in a way that takes both solutions into account? This is the question I address in this paper.

I introduce an axiom, *Kalai-Smorodinsky-Nash robustness* (KSNR, for short) that demands the following: in each problem, each player should receive at least the minimum of the payoffs he would have received under the Nash and under the Kalai-Smorodinsky solutions. The informal claim of this paper is that this axiom offers a reasonable notion of compromise between the two solutions.

The next section present the model, including all the necessary preliminaries. Next, Section 3 present the new axiom, KSNR, and the main results: characterizations of both solutions in terms of this axiom. The Nash solution in uniquely pinned down by KSNR and IIA and the Kalai-Smorodinsky solution—by KSNR and individual monotonicity. Thus, these results are not merely technical innovations: they provide a formal sense to a general intuition that has been with us, bargaining theorists, for quite some time: the Nash solution is associated with the idea of independence and the Kalai-Smorodinsky solution is associated with monotonicity.

The rest of the Sections are devoted to discussions about KSNR from various angles. Section 4 discusses its relation to the idea of interpersonal utility comparisons,

[2] To cite several references from a large literature: see Thomson (1994) on the cooperative approach to bargaining; see Rubinstein (1982) and Binmore, Rubinstein and Wolinsky (1986) on the noncooperative foundations of the Nash solution, and Moulin (1984) on the noncooperative foundations of the Kalai-Smorodinsky solution. The relevance of these solutions—and that of the Nash solution in particular—goes, even within economics, beyond the confines of theory; it is safe to say that it is not trivial, for example, to find a paper in which wage negotiations are considered and in which there is no reference to the "Nash product." Additionally, the bargaining model has found its way in recent years to completely new areas—computer science, electrical engineering, and more (see, e.g., Blocq and Orda (2012) and the references therein).

which is a surprising one, because the Nash and Kalai-Smorodinsky solutions are not "interpersonal." Section 5 discusses its relation to a certain class of solutions that was introduced by Sobel (2001), Section 6 considers the idea of preferences over bargaining solutions, and Section 7 considers the relation between KSNR and meta-bargaining. Section 8 concludes.

2 Model

A *bargaining problem* is defined as a pair (S, d), where $S \subset \mathbb{R}^2$ is the *feasible set*, representing all possible (v-N.M) utility agreements between the two players, and $d \in S$, the *disagreement point*, is a point that specifies their utilities in case they do not reach a unanimous agreement on some point of S. The following assumptions are made on (S, d):

- S is compact and convex;
- $d < x$ for some $x \in S$;[3]
- For all $x \in S$ and $y \in \mathbb{R}^2$: $d \leq y \leq x \Rightarrow y \in S$.

Denote by $\mathcal{B}$ the collection of all such pairs (S, d). A *solution* is any function $\mu \colon \mathcal{B} \to \mathbb{R}^2$ that satisfies $\mu(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. Given a feasible set S, the *weak Pareto frontier* of S is $WP(S) \equiv \{x \in S : y > x \Rightarrow y \notin S\}$ and the *strict Pareto frontier* of S is $P(S) \equiv \{x \in S : y \gneqq x \Rightarrow y \notin S\}$. The best that player i can hope for in the problem (S, d), given that player j obtains at least d_j utility units, is $a_i(S, d) \equiv \max\{x_i : x \in S_d\}$, where $S_d \equiv \{x \in S : x \geq d\}$. The point $a(S, d) = (a_1(S, d), a_2(S, d))$ is the *ideal point* of the problem (S, d). The *Kalai-Smorodinsky solution*, KS, due to Kalai and Smorodinsky (1975), is defined by

[3] Vector inequalities: xRy if and only if $x_i R y_i$ for both $i \in \{1, 2\}$, $R \in \{>, \geq\}$; $x \gneqq y$ if and only if $x \geq y$ & $x \neq y$.

$KS(S,d) \equiv P(S) \cap [d; a(S,d)]$.[4] The *Nash solution*, N, due to Nash (1950), is defined to be the unique maximizer of $(x_1 - d_1) \times (x_2 - d_2)$ over S_d.

Nash (1950) showed that N is the unique solution that satisfies the following axioms, in the statements of which (S,d) and (T,e) are arbitrary problems.

Weak Pareto Optimality (WPO): $\mu(S,d) \in WP(S)$.[5]

Individual Rationality (IR): $\mu_i(S,d) \geq d_i$ for all $i \in \{1,2\}$.

Let F_A denote the set of positive affine transformations from $\mathbb{R}$ to itself.[6]

Independence of Equivalent Utility Representations (IEUR): $f = (f_1, f_2) \in F_A \times F_A \Rightarrow f \circ \mu(S,d) = \mu(f \circ S, f \circ d)$.[7]

Let $\pi(a,b) \equiv (b,a)$.

Symmetry (SY): $[\pi \circ S = S] \& [\pi \circ d = d] \Rightarrow \mu_1(S,d) = \mu_2(S,d)$.

Independence of Irrelevant Alternatives (IIA): $[S \subset T] \& [d = e] \& [\mu(T,e) \in S] \Rightarrow \mu(S,d) = \mu(T,e)$.

Whereas the first four axioms are widely accepted, criticism has been raised regarding IIA. The idea behind a typical such criticism is that the bargaining solution could, or even should, depend on the shape of the feasible set. In particular, Kalai and

[4] Given two vectors x and y, the segment connecting them is denoted $[x; y]$.

[5] A natural strengthening of this axiom is Pareto Optimality (PO), which requires $\mu(S,d) \in P(S)$ for all $(S,d) \in \mathcal{B}$.

[6] i.e., the set of functions f of the form $f(x) = \alpha x + \beta$, where $\alpha > 0$.

[7] If $f_i \colon \mathbb{R} \to \mathbb{R}$ for each $i = 1, 2$, $x \in \mathbb{R}^2$, and $A \subset \mathbb{R}^2$, then: $(f_1, f_2) \circ x \equiv (f_1(x_1), f_2(x_2))$ and $(f_1, f_2) \circ A \equiv \{(f_1, f_2) \circ a : a \in A\}$.

Smorodinsky (1975) noted that when the feasible set expands in such a way that for every feasible payoff for player 1 the maximal feasible payoff for player 2 increases, it may be the case that player 2 loses from this expansion under the Nash solution. Given $x \in S_d$, let $g_i^S(x_j)$ be the maximal possible payoff for i in S given that j's payoff is x_j, where $\{i, j\} = \{1, 2\}$. What Kalai and Smorodinsky noted, is that N violates the following axiom, in the statement of which (S, d) and (T, d) are arbitrary problems with a common disagreement point.

Individual Monotonicity (IM):

$$[a_j(S,d) = a_j(T,d)]\&[g_i^S(x_j) \leq g_i^T(x_j)\ \forall x \in S_d \cap T_d] \Rightarrow \mu_i(S,d) \leq \mu_i(T,d).$$

Furthermore, they showed that when IIA is deleted from the list of Nash's axioms and replaced by IM, a characterization of KS obtains.[8] Both solutions have justifiably received an enormous amount of attention in the literature—in game theory and economics as well as in other fields.

3 Kalai-Smorodinsky-Nash robustness

Consider the following axiom, in the statement of which (S, d) is an arbitrary problem and i is an arbitrary player.

Kalai-Smorodinsky-Nash Robustness (KSNR): $\mu_i(S,d) \geq \min\{N_i(S,d), KS_i(S,d)\}$.

Theorem 1. *The Nash solution is the unique solution that satisfies Kalai-Smorodinsky-Nash robustness and independence of irrelevant alternatives.*

Theorem 2. *The Kalai-Smorodinsky solution is the unique solution that satisfies Kalai-Smorodinsky-Nash robustness and individual monotonicity.*

[8] In fact, IR is redundant: the remaining four axioms suffice for the characterization.

The proof of Theorems 1 and 2 follows from the combination of two results from the existing literature. The first of these results is based on the following axiom, in the statement of which (S, d) is an arbitrary problem.

Midpoint Domination (MD): $\mu(S, d) \geq \frac{1}{2}d + \frac{1}{2}a(S, d)$.

In words, MD requires the solution to Pareto dominate "randomized dictatorship." MD is a weakening of KSNR, because both N and KS satisfy MD.[9]

The following axiom, which is due to Anbarci (1998), has a similar flavor to that of MD, but the two are not logically comparable.

Balanced Focal Point (BFP): If $S = d + \text{conv hull}\{\mathbf{0}, (a, b), (\lambda a, 0), (0, \lambda b)\}$ for some $\lambda \in [1, 2]$, then $\mu(S, d) = d + (a, b)$.[10]

Note that for the particular value $\lambda = 2$, BFP becomes the requirement that MD be satisfied on triangles. The justification for this axiom is that the equal areas to the north-west and south-east of the focal point $d + (a, b)$ can be viewed as representing equivalent concessions. It is easy to see that, similarly to MD, BFP is implied by KSNR.[11]

Anbarci (1998) showed that KS is characterized by IM and BFP. His work was inspired by that of Moulin (1983), who in what is probably the simplest and most elegant axiomatization of N, proved that it is the unique solution that satisfies IIA and MD. Combining the results of Moulin (1983), Anbarci (1998), and the implication KSNR$\Rightarrow$ [MD, BFP], one obtains a proof of the theorems.

[9]That KS satisfies MD is obvious; the fact that N satisfies it was proved by Sobel (1981).

[10]Anbarci assumes the normalization $d = \mathbf{0} \equiv (0, 0)$; the version above is the natural adaptation of his axiom to the model with an arbitrary d.

[11]It is easy to see that if S satisfies the conditions specified in BFP, and μ is a solution that satisfies WPO, SY, and IEUR, then $\mu(S, d) = d + (a, b)$.

4 KSNR and interpersonal comparisons of utility

Three major solutions capture, in different ways, *interpersonal utility comparisons*. The *egalitarian solution*, E, due to Kalai (1977), is defined by $E(S,d) \equiv d + (g,g)$, where g, standing for "gain," is the maximal number such that the aforementioned expression is in S; the *equal-loss solution*, EL, due to Chun (1988), is defined by $EL(S,d) \equiv a(S,d) - (l,l)$, where l, standing for "loss," is the minimal number such that the aforementioned expression is in S; a solution is *utilitarian* if it is a selection from $\mathbb{U}(S,d) \equiv \text{argmax}_{S_d}\Sigma x_i$. The solutions E and EL compare utilities in obvious ways, and the $\mathbb{U}$ expresses the idea that an agreement x is better than an agreement y if when the former replaces the latter player i only "loses a little bit," but player j "gains a lot."

Both N and KS, on the other hand, satisfy IEUR, and therefore exclude interpersonal comparisons. Nonetheless, KSNR (without the further imposition of IIA or IM) allows for such considerations. This is seen, for example, in the "KSNR version" of the egalitarian solution, $\mu^E(S,d) \equiv \min\{N(S,d), KS(S,d)\} + (g,g)$, where g is the maximal number such that the aforementioned expression is in S.

As surprisingly turns out, KSNR not only allows for interpersonal considerations, but, in an indirect way, assumes them. This way is described below, in Theorem 3. To state the theorem, the following definition is needed.

Interpersonal Comparisons Robustness (ICR): There exists a selection from $\mathbb{U}$, U, such that $\mu_i(S,d) \geq \min\{U_i(S,d), E_i(S,d), EL_i(S,d)\}$ for all i and (S,d).

Theorem 3. *Kalai-Smorodinsky-Nash robustness implies interpersonal comparisons robustness.*

The bound $\mu(S,d) \geq \min\{U(S,d), E(S,d), EL(S,d)\}$ can be interpreted as a compromise between fairness and efficiency. Since KSNR implies ICR, it can be thought of as a reconciliation of fairness and efficiency which (a) is sufficiently rich as to take into account E, U, and EL, and (b) is not too restrictive, in the sense that it does not exclude IEUR.

5 KSNR and normalized CES solutions

Sobel (2001) studied bargaining in an allocation-problem context, where n agents need to divide among themselves m commodities.[12] The allocation is to be carried out with the help of a social planner, with whom the agents play the following *distortion game*: they simultaneously report utility functions to the planner (or misreport, hence the game's name), who constructs a bargaining problem from these reports: the utility image of the set of feasible allocations under the reported utilities. Next, he chooses an allocation whose utility image is consistent with a pre-specified bargaining solution. In this paper, Sobel focuses on the distortion game induced by the *relative utilitarian solution*, *RU*—the solution that picks a maximizer of a weighted sum of the players' utilities, where a player's weight equals the inverse of the distance between his maximum and minimum feasible utilities.[13] In an earlier paper, Sobel (1981), he studied essentially the same game, but for bargaining solutions that, as opposed to RU, satisfy MD.

Despite the differences between the two settings, it turns out that certain type of allocations—*constrained equal-income competitive equilibrium allocations*[14]—emerge as Nash equilibria in either case. This lead Sobel to look closely at a family of solutions, indexed by a single parameter $a \leq 1$, which includes, as special cases, the Nash and Kalai-Smorodinsky solutions, on which he focused in his 1981 paper, and the relative utilitarian solution, on which he focused in his 2001 paper. The

[12] In what follows, I consider $n = 2$.

[13] This solution was proposed by Cao (1982) for the bargaining problem; subsequent works—for example, Dhillon and Mertens (1999), Karni and Safra (2002), Karni (1998), and Segal (2000)—study this solution in a social choice context.

[14] See Sobel (1981, 2001) for definition and discussion.

solutions in this family are defined on *normalized problem*—(S,d) for which $d = \mathbf{0}$ and $a(S,d) = (1,1)$—which, for Sobel, is without loss of generality, because in either paper he only considers IEUR solutions.

Given $a \leq 1$, the corresponding solution is

$$W(S,a) \equiv \arg\max_{x \in S} [\frac{1}{2} x_1^a + \frac{1}{2} x_2^a]^{\frac{1}{a}}.$$

The maximizer is unique for $a < 1$. For $a = 1$, which corresponds to the relative utilitarian solution, any symmetric selection from the set of maximizers is allowed. The Nash solution corresponds to $\lim_{a \to 0} W(.,a)$ and the Kalai-Smorodinsky solution corresponds to $\lim_{a \to -\infty} W(.,a)$. In light of the resemblance to the well-known concept from Consumer Theory, I will call these solutions *normalized CES solutions.*

Interestingly, on the family of normalized CES solutions, KSNR simply becomes the requirement that the solution Pareto dominate "randomized dictatorship."

Theorem 4. *A normalized CES solution satisfies midpoint domination if and only if it satisfies Kalai-Smorodinsky-Nash robustness.*

6 Preferences over solutions

The premise of this paper is that both the Nash and Kalai-Smorodinsky solutions are "good." In other words, implicit here is the assumption of *preferences over solutions*—some solutions are better than others. Let us formalize this idea.

Consider an arbitrator who has a preference relation (i.e., a complete, transitive, reflexive binary relation) over the set of all solutions. Let Σ denote the set of the maximizers of the arbitrator's preferences. Namely, Σ is the set of solutions that the arbitrator finds most appealing.

Given two solutions, μ and ν, and a collection of bargaining problems, $\tilde{\mathcal{B}}$, define the solution $\phi = \phi(\mu, \nu, \tilde{\mathcal{B}})$ by

$$\phi(S,d) \equiv \begin{cases} \mu(S,d) & \text{if } (S,d) \in \tilde{\mathcal{B}} \\ \nu(S,d) & \text{otherwise} \end{cases}$$

Say that a set of solutions, Σ_0, is *closed*, if for every $\mu, \nu \in \Sigma_0$ and every $\tilde{\mathcal{B}} \subset \mathcal{B}$, it is true that $\phi(\mu, \nu, \tilde{\mathcal{B}}) \in \Sigma_0$.

Now, if μ and ν are two solutions that the arbitrator views as optimal, then there is nothing to stop him from employing one of them in some circumstance and employing the other in other circumstances. In other words, it is sensible to assume that the set of the arbitrator's optima, Σ, is closed. From here, just one additional step leads to KSNR.

Theorem 5. *Let Σ be a closed set of solutions. Suppose that the Nash solution is the unique element of Σ that satisfies independence of irrelevant alternatives. Then Σ contains only KSNR solutions.*

7 Meta-bargaining

The idea of preferences over solutions is closely related to the idea of *bargaining* over solutions: if the two players have different views as to what is a "good" solution, one may naturally consider the possibility that they bargain over which solution to employ. This idea, which is called in the literature *meta bargaining*, originated in a 1986 paper of van Damme, who considered the following noncooperative 2-person game.

First, each player announces a bargaining solution from a certain set of allowable solutions. If the players' demands are jointly feasible, then the demanded payoffs are implemented, and the game is over; otherwise, the feasible set is truncated as to include only payoffs that adhere to the following restriction: if player i demanded for himself x_i, and (u_1, u_1) is in the truncated set, then $u_i \leq x_i$. The solutions are then applied to the truncated set, and the procedure repeats itself iteratively, to a limit. If the two limits coincide, the common point is the solution; otherwise—i.e., in the case of "perpetual disagreement"—each player receives the disagreement payoff, which is normalized to zero. Both players immediately agreeing on the Nash solution is a Nash equilibrium of this game, and, moreover, every Nash equilibrium payoff vector coincides with the one of the Nash bargaining solution.

van Damme's paper generated a strand of follow-up papers on meta bargaining (e.g., Naeve-Steinweg (1999), Trockel (2002)), from which, the closest one to the work presented here, is a paper by Anbarci and Yi (1992). They propose the following dynamic procedure, called the *Minimal Agreement Procedure*.[15] Given a bargaining problem (S, d), it is defined as follows. First, each player i choses a strategy: an MD-satisfying solution, μ^i; if $(\mu_1^1(S, d), \mu_2^2(S, d)) \in S$ then the process terminates, with the aforementioned point being the solution. Otherwise, a new problem is considered, $(S, (\min(\mu_1^1(S, d), \mu_1^2(S, d)), \min(\mu_2^1(S, d), \mu_2^2(S, d))))$. The process then repeats itself iteratively. Anbarci and Yi proved that this process converges to a unique limit. Since KSNR implies MD, it follows that if the players play this game and strategy sets are confined to KSNR solutions, then a unique outcome is guaranteed.

8 Closing comments

I have introduced the notion of Kalai-Smorodinsky-Nash Robustness (KSNR) and showed that N is the unique such solution that satisfies IIA and KS is the unique such solution that satisfies IM. I have also introduced the notion of a closed set of bargaining solutions, and showed that under closedness, KSNR is the weakest axiom under which the aforementioned identification ($N \sim$ IIA and $KS \sim$ IM) obtains.[16]

The bargaining model, that has been considered here for the case of two players, has a well-known n-player counterpart. Theorems 1, 2, and 5 generalize straight-

[15] A similar procedure has been studied by Marco et al. (1995).

[16] In fact (as we saw in Theorem 5), the identification $N \sim IIA$ is sufficient for this conclusion.

forwardly to the $n \geq 3$ case.[17] The result about interpersonal utility comparisons, Theorems 3, and the result about normalized CES solutions, Theorem 4, do not.[18]

The general theme that has been introduced here is the "merging" of bargaining solutions. This idea, to the best of my knowledge, is a novelty. The justification for "merging" lies in the assumption, that there are multiple solutions that are viewed as best; the underlying objects that motivate the analysis, therefore, are preferences over solutions. A more traditional approach to the study of such preferences is the decision-theory-type approach: to postulate axioms on these preferences, and see what conclusions can be drawn. This line of research has been taken by Border and Segal (1997), who showed that if these preferences satisfy certain appealing axioms, then the Nash solution is best according to them. It would be interesting to obtain a result of this kind for the Kalai-Smorodinsky solution.

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Appendix

Theorem 3 follows from the combination of the following lemmas.

Lemma 1. *$KS_i(S,d) \geq min\{E_i(S,d), EL_i(S,d)\}$ for all (S,d) and i.*

Proof. Let (S,d). Let $a \equiv a(S,d)$, let $k \equiv KS(S,d)$, and let $x \equiv EL(S,d)$. Wlog, suppose that $d = (0,0)$. If $a_1 = a_2$ then $k = E(S,d) = EL(S,d)$ and we are done. Suppose then, wlog, that $a_1 > a_2$. Obviously, we can also assume, wlog, that $a_1 = 1$.

[17] The results of Moulin (1983) and Anbarci (1998) that were cited in Section 3 generalize to an arbitrary number of players and therefore so do Theorems 1 and 2. The proofs of the n-player version of Theorem 5 is identical to the one from the 2-player case.

[18] The equal-loss solution is not even well-defined in the $n \geq 3$ case (it is well-defined in this case if the feasible sets are assumed to be comprehensive, not compact). As for the nonextendability of Theorem 4, the Appendix provides a counterexample.

In this case $k_1 > E_1(S,d)$; I will prove that $k_2 \geq x_2$. Assume by contradiction that $k_2 < x_2$. Since $\frac{k_2}{k_1} = \frac{a_2}{a_1} = a_2$ we have $k_2 = a_2 k_1$. Therefore $a_2 k_1 < x_2$. By the definition of EL, $a_1 - x_1 = a_2 - x_2$, hence $x_2 = a_2 + x_1 - a_1 = a_2 + x_1 - 1$. Therefore $a_2 + x_1 - 1 > a_2 k_1 \Rightarrow a_2(1-k_1) > 1 - x_1$, and since $a_2 < a_1 = 1$, $1 - k_1 > 1 - x_1$, or $x_1 > k_1$. We obtain that $x_i > k_i$ for both $i \in \{1,2\}$, in contradiction to PO.

□

Lemma 2. *There exists a selection $U \in \mathbb{U}$, such that $N_i(S,d) \geq min\{E_i(S,d), U_i(S,d)\}$, for all (S,d) and i.*

Proof. In Rachmilevitch (2011) I proved this result for the domain of *strictly comprehensive problems*—those whose feasible set (is a d-translation of a set that) takes the form $S = \{(a, f(a)) : a \in [0, A]\}$ for some $A > 0$, where f is a differentiable concave function.[19] Let U' be the selection defined on this domain. Since the correspondence $\mathbb{U}$ is upper hemi continuous, the selection we seek, $U(S,d)$, can be obtained as follows: on a strictly comprehensive (S,d) it coincides with $U'(S,d)$, and otherwise it is defined to be by an arbitrary limit of $U'(S_n,d)$, where $\{S_n\}$ is some sequence of feasible sets that converges to S in the Hausdorff topology. □

Proof of Theorem 4: By Sobel (2001), we know that MD is equivalent to $a \leq 0$. I will therefore prove that KSNR is also equivalent to $a \leq 0$. Moreover, it is enough to prove this equivalence for $a < 0$ (because $a = 0$ corresponds to the Nash solution). For simplicity (and without loss), I will consider only (normalized) strictly comprehensive problems (see Lemma 2 above for their definition); the case of an arbitrary problem then follows from standard limit arguments.

The parameter a corresponds to the solution

$$W(S,a) \equiv \arg\max_{x \in S_0} [\frac{1}{2}x_1^a + \frac{1}{2}x_2^a]^{\frac{1}{a}}.$$

[19]This result is closely related to famous theorems of Harsanyi (1959) and Shapley (1969). See Rachmilevitch (2011) for details.

In the case of a normalized strictly comprehensive problem, the object of interest is therefore

$$\text{arg max}_{0\leq x\leq 1}[\frac{1}{2}x^a + \frac{1}{2}f(x)^a]^{\frac{1}{a}} \equiv W(x,a).$$

Therefore $\frac{d}{dx}W(x,a) = e[\frac{a}{2}x^{a-1} + \frac{a}{2}f(x)^{a-1}f'(x)]$, where e is a shorthand for a strictly positive expression.

At the optimum, this derivative is either zero or positive. Suppose first that it is positive. Namely, $x^{a-1} > -f(x)^{a-1}f'(x)$ at $x = 1$, or

$$f(1)^{1-a} > -f'(1).$$

Since $a < 0$, this implies $f(1) > -f'(1)$, therefore $(a, f(1)) = N(S, \mathbf{0})$. The solution payoff coincides with the one recommended by the Nash solution, hence KSNR holds.

Now consider the case where the first order condition holds. This condition is

$$[\frac{f(x(a))}{x(a)}]^{1-a} = -f'(x(a)),$$

where $x(a)$ is player 1's solution payoff given the parameter a. The derivative of the RHS with respect to a is $-f''(x(a))x'(a)$, hence the sign of $x'(a)$ is the same as the sign of the derivative of the LHS. To compute the latter, recall the formula

$$\Psi'(a) = \Psi(a) \times \{h'(a)\text{log}[g(a)] + \frac{h(a)g'(a)}{g(a)}\},$$

where $\Psi(a) \equiv [g(a)]^{h(a)}$.

Taking $g(a) \equiv \frac{f(x(a))}{x(a)}$ and $h(a) \equiv 1 - a$, we see that the signs of the derivative of the LHS is the same as the sign of

$$-\text{log}[\frac{f(x(a))}{x(a)}] + \frac{(1-a)}{f(x(a))x(a)}x'(a)(f'(x(a))x(a) - f(x(a))) \equiv -\text{log}[\frac{f(x(a))}{x(a)}] + Zx'(a),$$

where Z is a shorthand for a negative expression. Therefore, the sign of $x'(a)$ is the same as that of $-\text{log}[\frac{f(x(a))}{x(a)}] + Zx'(a)$.

Let $k \equiv KS_1(S, \mathbf{0})$ and $n \equiv N_1(S, \mathbf{0})$. If the parameter a is such that $x(a) = k$, we are done. Suppose then that $x(a) \neq k$.

Case 1: $x(a) < k$. In this case, $\frac{f(x(a))}{x(a)} > 1$, hence $-\log[\frac{f(x(a))}{x(a)}] < 0$. This means that $x'(a) < 0$. To see this, assume by contradiction that $x'(a) \geq 0$. This means that $\text{sign}[-\log[\frac{f(x(a))}{x(a)}] + Zx'(a)] = -1 = \text{sign}[x'(a)]$, a contradiction. Therefore $n < x(a) < k$, so KSNR holds.

Case 2: $x(a) > k$. In this case, $\frac{f(x(a))}{x(a)} < 1$, hence $-\log[\frac{f(x(a))}{x(a)}] > 0$. This means that $x'(a) > 0$. To see this, assume by contradiction that $x'(a) \leq 0$. This means that $\text{sign}[-\log[\frac{f(x(a))}{x(a)}] + Zx'(a)] = 1 = \text{sign}[x'(a)]$, a contradiction. Therefore $k < x(a) < n$, so KSNR holds. $\square$

Proof of Theorem 5: Make the assumptions of the theorem and assume by contradiction that there is a solution $\mu \in \Sigma_0$ such that $\mu_i(S^*, d^*) < \min\{N_i(S^*, d^*), KS_i(S^*, d^*)\}$ for some (S^*, d^*) and i. Define the solution ν by

$$\nu(S, d) \equiv \begin{cases} \mu(S, d) & \text{if } \mu(S^*, d^*) \in S \text{ and } d = d^* \\ N(S, d) & \text{otherwise} \end{cases}$$

By closedness, $\nu \in \Sigma$. Additionally, ν satisfies IIA, but $\nu \neq N$.[20] $\square$

I conclude with a 3-person example that shows that Theorem 4 does not generalize to the multi-player case. Let $S \equiv \text{conv hull}\{(1, 0, \frac{2}{3}), (0, 1, \frac{2}{3}), (1, 1, 0), (0, 0, 1)\}$. We have that $N(S, \mathbf{0}) = KS(S, \mathbf{0}) = (\frac{1}{2}, \frac{1}{2}, \frac{2}{3})$. So, we only need to find some parameter $a < 0$ given which the corresponding solution selects a different point. The objective to be maximized is $[\frac{1}{3}x^a + \frac{1}{3}x^a + \frac{1}{3}f(x)^a]$, where $f(x)$ describes the maximum feasible payoff for player 3, given that players 1 and 2 obtain the identical payoffs x. To find $f(x)$,

[20] The satisfaction of IIA is easy to see. In fact, it is clear that it only needs to be proved for the case where $d = d^*$. Look at (T, d^*) and (S, d^*) where $S \subset T$ contains $x \equiv \mu(T, d^*)$. If $x = \mu(S^*, d^*)$, then, by definition of ν, $\nu(S, d^*) = \nu(T, d^*) = x$. If, on the other hand, $x \neq \mu(S^*, d^*)$ then $\mu(S^*, d^*) \notin T$ and hence not in S; in this case, $x = N(T, d^*) = N(S, d^*) = \nu(S, d^*)$.

note that any point on the Pareto frontier takes the form $(\alpha+\beta, \alpha+\beta, 1-\frac{2}{3}\alpha-\beta)$, $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$. Thus, to find $f(x)$ we need to maximize $1-\frac{2}{3}\alpha-\beta$ subject to $\alpha+\beta = x$. The solution obtains at $\beta = 0$, hence the objective to be maximized is $[\frac{1}{3}x^a + \frac{1}{3}x^a + \frac{1}{3}(1-\frac{2}{3}x)^a]$. The first order condition implies $\frac{2a}{3}x^{a-1} = \frac{2a}{9}(1-\frac{2}{3}x)^{a-1}$, or $3(1-\frac{2}{3}x)^{1-a} = x^{1-a}$. For $a = -\frac{1}{2}$, this equation does not hold when $x = \frac{1}{2}$.

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