The Nash solution is more efficient than fair

Shiran Rachmilevitch*

July 15, 2013

Abstract

I state and prove two formal versions of the claim that even though the Nash (1950) bargaining solution offers a certain reconciliation of fairness and efficiency considerations (egalitarianism vs. utilitarianism), it puts more emphasis on the latter rather than on the former.

Keywords: Bargaining; efficiency; egalitarianism; fairness; utilitarianism.
JEL Codes: D63; D71

1 Introduction

Nash's (1950) *bargaining problem* is described in the utility space: two players face a compact and convex set $S \subset \mathbb{R}^2_+$ of available utility pairs, from which they need to agree on one; if they fail to reach an agreement, both get zero.[1]

*Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. Tel: +972-4-8249588; fax: +972-4-8240059. E-mail address: shiranrach@econ.haifa.ac.il

[1] It is assumed that $\mathbf{0} \equiv (0,0) \in S$ for every bargaining problem S, so zero payoffs are always feasible; also, it is assumed that there is an $x \in S$ with $x > \mathbf{0}$, so cooperation is worthwhile.

The *Nash solution* (1950) to the problem S, $N(S)$, is the maximizer of $x_1 \times x_2$ over $x \in S$ (this product is called the *Nash product*). In general, a *bargaining solution* is a *selection*—a function that picks a unique point of S for every such S.

Two other known solutions—ones that express fundamental and competing principles of distributive justice—are the *egalitarian solution*, $E(S)$, and the *utilitarian solution*, $U(S)$. Like the Nash solution, they are both defined in terms of maximization: the former maximizes $\min\{x_1, x_2\}$ over $x \in S$ and the latter maximizes $x_1 + x_2$ over $x \in S$.[2]

In the bargaining framework, the Nash solution offers a simple compromise between egalitarianism and utilitarianism: for any S, the point $N(S)$ lies on S's boundary in between $E(S)$ and $U(S)$. To the best of my knowledge, this result has been proved for the first time by Cao (1982). I have recently studied this relationship among the three solutions in Rachmilevitch (2013b); as I have pointed out in that paper, this relationship is related to the fact that the Nash solution is the only solution that jointly satisfies the egalitarian and utilitarian objectives for some rescaling of the individual utilities (see Harsanyi (1959) and Shapley (1969)).

It is natural to ask which side in the compromise between egalitarianism and utilitarianism, if any, is being favored by the Nash solution. Below I prove results that formalize the idea that the Nash solution is "at least as efficient as it is fair." The results are given below in Sections 2 and 3. Whereas the result of Section 3 is novel, the results in Section 2 are formal versions of some well-known facts about the geometry of the Nash solution; however, as far as I know, they have not been previously published.

[2] E was axiomatized for the first time by Kalai (1977). U, in general, is multi-valued; in this short paper I will only consider problems for which it is single-valued.

2 Fairness implies efficiency, but not vice versa

Let $\mathcal{B}$ denote the collection of bargaining problems S such that $U(S)$ is single-valued.

Proposition 1. *Let $S \in \mathcal{B}$ be such that $N(S) = E(S)$. Then $N(S) = U(S)$.*

Proof. Let S be a problem and assume by contradiction that $N(S) = E(S) = (e, e) \neq U(S) = (x, y)$. By the uniqueness of the Nash product's maximizer, $e^2 > s_1 \times s_2$ for any $s \in S \setminus \{(e, e)\}$. By definition of U, $x + y \geq 2e$. For each $\lambda \in [0, 1]$ consider $\lambda(e, e) + (1 - \lambda)(x, y) = (\lambda e + (1 - \lambda)x, \lambda e + (1 - \lambda)y)$, and let g denote the associated Nash product:

$$g(\lambda) \equiv \lambda^2 e^2 + \lambda(1 - \lambda)e(x + y) + (1 - \lambda)^2 xy.$$

Note that $g'(\lambda)|_{\lambda=1} > 0$. However, $g'(\lambda)|_{\lambda=1} = 2e^2 - e(x + y)$, so $g'(\lambda)|_{\lambda=1} > 0$ implies $2e > x + y \geq 2e$, a contradiction. $\square$

The "converse" of Proposition 1 is not true: the fact that N coincides with U on a certain problem does not imply that it also coincides with E on that problem. To see this, look at $S^* = \{\mathbf{0}, (\frac{3}{4}, \frac{3}{4}), (1, 0), (1-5\epsilon, 0.7+\epsilon)\}$; for a sufficiently small $\epsilon > 0$ we have $N(S^*) = U(S^*) = (1 - 5\epsilon, 0.7 + \epsilon) \neq (\frac{3}{4}, \frac{3}{4}) = E(S^*)$. This counterexample S^* enjoys two nice features. First, it is *normalized*; a problem S is *normalized* if $a_1(S) \equiv \max\{x : (x, y) \in S\} = a_2(S) \equiv \max\{y : (x, y) \in S\} = 1$. Second, its *weak Pareto frontier* coincides with its *strict Pareto frontier*. Normalized problems are especially appealing when the discussion is centered around the Nash solution because this solution is *scale invariant*, meaning that when independent linear transformations are applied to a given problem, the solution point of this problem varies in accordance to these transformations. Since the utility scales have no meaning in terms of comparisons to one another, the $0-1$ normalization is natural. The coincidence of the weak and strict frontiers is appealing since it expresses the idea that whenever one player is willing to compromise and give up something, the other player can gain something.[3]

Yet, there is—for lack of a better word—some "lack of elegance" in S^*: its boundary has "kinks," meaning that the "price" of one person's utility is terms of his partner's utility exhibits discontinuities. When this is ruled out—when, in addition to normalization, only *smooth* problems are considered—then a "converse" of Proposition 1 obtains.

Formally, let $\mathcal{B}^*$ be the set of *smooth normalized problems*—those $S \in \mathcal{B}$ such that $S = \{(x, f(x)) : x \in [0,1]\}$, where f is some concave, strictly decreasing differentiable function with $f(0) = 1$ and $f(1) = 0$.

Proposition 2. *Let $S \in \mathcal{B}^*$ be such that $N(S) = U(S)$. Then $N(S) = E(S)$.*

Proof. Let $S \in \mathcal{B}^*$ be such that $N(S) = U(S)$. Let f be the function describing S's boundary. The point $U(S) \equiv (x, f(x))$ satisfies the first-order-condition $-f'(x) = 1$. By the first order condition associated with N, $-f'(x) = \frac{f(x)}{x}$. Therefore $N(S) = E(S)$. □

So, even though the Nash solution's efficiency is generally more common (or more frequent) than its fairness, on the restricted domain $\mathcal{B}^*$ the two coincide. Can we say that N is "more efficient than fair" even if we restricted our attention to $\mathcal{B}^*$? As we will see in the next Section, the answer (in a certain sense) is YES.

[3]Finding examples of problems S for which $N(S) = U(S)$ but $N(S) \neq E(S)$ is easy if non-normalized problems and/or problems with different weak and strict frontiers are considered; look, for example, at $S = \text{conv hull}\{\mathbf{0}, (0,1), (2,0), (2,1)\}$: $N(S) = U(S) = (2,1) \neq (1,1) = E(S)$.

3 Betweenness

One of the simplest ways to compromise between egalitarianism and utilitarianism is to consider the average of their respective objectives: to maximize $\frac{1}{2}(x_1+x_2)+\frac{1}{2}\min\{x_1,x_2\}$ over $x \in S$. Let A denote this bargaining solution. When attention is restricted to the domain $\mathcal{B}^*$, the Nash solution is not only between E and U (this is true on the entire $\mathcal{B}$), but, moreover, it is between A and U—it is "closer to utilitarianism than to egalitarianism."[4] For proving this result, the following lemma will be useful.

Lemma 1. *Let $S \in \mathcal{B}^*$ and let $(x,y)=A(S)$. Then $x \in (0,1)$.*

Proof. Make the aforementioned assumptions, and assume by contradiction that $x \in \{0,1\}$. Let f be the function describing S's boundary and let $(e,e) \equiv E(S)$.

Case 1: $x=1$. In this case A maximizes $R(t) \equiv \frac{1}{2}[t+f(t)]+\frac{1}{2}f(t) = \frac{1}{2}t+f(t)$ over $(e,1]$. Since $x=1$, $R'(1) \geq 0$, or $\frac{1}{2} \geq -f'(1)$, which is impossible for a normalized problem.

Case 2: $x=0$. In this case A maximizes $L(t) \equiv \frac{1}{2}[t+f(t)]+\frac{1}{2}t = t+\frac{1}{2}f(t)$ over $[0,e)$. Since $x=0$, $L'(0) \leq 0$, or $1+\frac{1}{2}f'(0) \leq 0$, or $2 \leq -f'(0)$; again, this is impossible for a normalized problem. $\square$

Proposition 3. *Let $S \in \mathcal{B}^*$. Then the following is true for each $i \in \{1,2\}$:*

$$min\{A_i(S),U_i(S)\} \leq N_i(S) \leq max\{A_i(S),U_i(S)\}.$$

That is, the Nash solution, N, is "in between" A and U.

[4] I will prove this result for the domain $\mathcal{B}^*$, but it will be easy to see that it holds for any normalized problem for which U is single-valued; i.e., the boundary does not have to be smooth.

Proof. Let $S \in \mathcal{B}^*$. Let f be the function describing its boundary, let $(x, f(x)) \equiv A(S)$ and let $(e, e) \equiv E(S)$. If $(x, f(x)) = E(S)$ then we are done, since N is always between E and U. Suppose then that $(x, f(x)) \neq E(S)$.

Case 1: $x > f(x)$. In this case, A maximizes the objective function $R(t) \equiv \frac{1}{2}(t + f(t)) + \frac{1}{2}f(t) = \frac{1}{2}t + f(t)$ over $t \in (e, 1]$. $R'(t) = \frac{1}{2} + f'(t)$. By Lemma 1 the solution is interior and therefore $R'(x) = 0$, or $f'(x) = -\frac{1}{2}$; so $A(S)$ is to the left of $U(S)$. If $N(S) \equiv (n, f(n))$ is to the left of $A(S)$, then $-f'(n) < \frac{1}{2}$. By the first-order-condition associated with N, $-f'(n) = \frac{f(n)}{n}$,and so we would obtain $n > 2f(n)$; this, however, is impossible in a normalized problem, because $f(n) \geq \frac{1}{2}$.[5]

Case 2: $x < f(x)$. In this case, A maximizes the objective function $L(t) \equiv \frac{1}{2}(t + f(t)) + \frac{1}{2}t = t + \frac{1}{2}f(t)$ over $[0, e)$. $L'(t) = 1 + \frac{1}{2}f'(t)$. Here $L'(x) = 0$, or $f'(x) = -2$, so $A(S)$ is to the right of $U(S)$. If $N(S)$ is to the right of $A(S)$, then $\frac{f(n)}{n} > 2$, which contradicts midpoint domination. $\square$

4 Closing comments

The tension between egalitarianism and utilitarianism, which is a fundamental issue in collective decision making, has been dealt with extensively in the philosophy and economics literatures.[6] Additionally, in recent years research on the fairness-efficiency tradeoff has found its ways to new disciplines, such as computer science, engineering, and operations management;[7] indeed, this

[5]More generally, $N(S) \geq \frac{1}{2}a(S)$ for all S; this *midpoint domination* property of the Nash solution was proved by Sobel (1981).

[6]See, among many others, Arrow (1973), Harsanyi (1975), Rawls (1974), Sen (1974), and Yaari (1981).

[7]See, for example, Bertsimas et al (2011, 2012), Boche and Schubert (2009), and Gutman and Nisan (2012).

tradeoff is relevant to many areas of life.

Nash's (1950) solution is the main solution in the bargaining literature. Whereas previous works highlight the fairness aspects of this solution,[8] in this paper I have tried to emphasize its efficiency. It is interesting to view this efficiency in light of the fact that the Nash solution enjoys axiomatizations in which Pareto efficiency is not invoked at all.[9]

Finally, it is worth pointing out to the fact that the entire analysis in this paper has been carried out in the "traditional" framework of 2-person convex bargaining problems. Deviations from this framework stress even further that the Nash solution is "more efficient than fair." In this context, it is helpful to consider the *Kalai-Smorodinsky solution*, KS (Kalai and Smorodinsky (1975)) which, alongside Nash's, is the literature's main scale invariant solution. For any problem S, $KS(S) = \theta a(S)$, where θ is the maximal number such that this expression is in S. For any $n \in \mathbb{N}$, the n-person bargaining problem is defined in an analogous way to the 2-person problem, and the definitions of N and KS extend straightforwardly to this case. However, for $n \geq 3$ the Kalai-Smorodinsky solution violates weak Pareto efficiency,[10] whereas the Nash solution, by contrast, satisfies strong Pareto optimality. For *non-convex* 2-person problems, there may exist multiple maximizers to the Nash product, some of which may exhibit high degree of inequality among the players; the Kalai-Smorodinsky solution, by contrast, is always single-valued, and always "normalized egalitarian."

[8]See, for example, Anbarci and Sun (2011b), Mariotti (1999), and Trockel (2005).

[9]Roth (1977) showed that efficiency can be deleted from Nash's (1950) classic axiomatization if *strong individual rationality* is added; more recent strengthenings and variants of Roth's result have been obtained by Anbarci and Sun (2011a) and Rachmilevitch (2013a).

[10]See Roth (1979).

References

[1] Anbarci, N., and Sun, C.J. (2011a), Weakest collective and the Nash bargaining solution, *Social Choice and Welfare*, **37**, 425-429.

[2] Anbarci, N., and Sun, C.J. (2011b), Distributive justice and the Nash bargaining solution, *Social Choice and Welfare*, **37**, 453-470.

[3] Arrow, K.J. (1973), Some ordinalist-utilitarian notes on Rawls' Theory of Justice, *The Journal of Philosophy*, **70**, 245-263.

[4] Bertsimas, D., Farias V., and Trichakis, N. (2011), The price of fairness, *Operations Research*, **59**, 17-31.

[5] Bertsimas, D., Farias V., and Trichakis, N. (2012), On the efficiency-fairness trade-off, *Management Science*, **58**, 2234-2250.

[6] Boche, H., and Schubert, M. (2009), Nash bargaining and proportional fairness for wireless systems, *IEEE/ACM Transactions on Networking*, **17**.

[7] Cao, X. (1982), Performance functions and bargaining solution, in the proceedings of the 21st IEEE Conference on Decision and Control, Orlando, FL, USA, December 1982.

[8] Gutman, A., and Nisan, N. (2012), Fair allocation without trade, AAMAS, Valencia, Spain, June 2012.

[9] Harsanyi, J,C. (1959), A bargaining model for the cooperative n-person games. In: Tucker AW, Luce RD (eds) Contributions to the theory of Games IV. Princeton University Press, Princeton, 325-355.

[10] Harsanyi, J.C. (1975), Can the maxmin principle serve as a basis for morality? *The American Political Science Review*, **69**, 594-606.

[11] Kalai, E. (1977), Proportional solutions to bargaining situations: interpersonal utility comparisons, *Econometrica*, **45**, 1623-1630.

[12] Kalai, E., and Smorodinsky, M. (1975), Other solutions to Nash's bar-

gaining problem, *Econometrica*, **43**, 513-518.

[13] Mariotti, M. (1999), Fair bargains: Distributive justice and Nash bargaining theory, *Review of Economic Studies*, **66**, 733-741.

[14] Nash, J.F. (1950), The bargaining problem, *Econometrica*, **18** ,155-162.

[15] Rawls, J. (1974), Some reasons for the maxmin Criterion, *The American Economic Review*, **64**, 141-146.

[16] Rachmilevitch, S. (2013a), Disagreement in bargaining and the Nash solution, working paper.

[17] Rachmilevitch, S. (2013b), Utilitarianism, egalitarianism, and the Nash bargaining solution, working paper.

[18] Roth, A.E. (1977), Individual rationality and Nash's solution to the bargaining problem, *Mathematics of Operations Research*, **2**, 64-65.

[19] Roth, A.E., (1979), An impossibility result concerning n-person bargaining games, *International Journal of Game Theory*, **8**, 129-132.

[20] Sen. A.K. (1974), Rawls versus Bentham: An axiomatic examination of the pure distribution problem, *Theory and Decision*, **4**, 301-309.

[21] Shapley, L.S., (1969), "Utility Comparison and the Theory of Games." In: *La Décision: Agrégation et Dynamique des Ordres de Préférence,* Editions du CNRS, Paris, 251-263.

[22] Sobel .J. (1981), Distortion of utilities and the bargaining problem, *Econometrica*, **49**, 597-619.

[23] Trockel, W. (2005), In what sense is the Nash solution fair? in: Schmidt, H., and Traub, S. (eds): Advances in public economics: Utility, choice, and welfare, Springer, Heidelberg.

[24] Yaari, M.E. (1981), Rawls, Edgeworth, Shapley, Nash: Theories of distributive justice re-examined, *Journal of Economic Theory*, **24**, 1-39.