Microeconomics MSc

Choice Under Uncertainty

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November 2010

Simple Lottery

- A simple lottery is an $L = (p_1, p_2, \ldots, p_n)$ where $p_1 + p_2 + \ldots + p_n = 1$ and $p_i \geq 0$ for all i.
- Probability p_i represents the probability that outcome i occurs.
- For example a coin is $p_1 = p_2 = 1/2$ (say 1 represents heads and 2 tails).
- A die is $p_1 = p_2 = .. = p_6 = 1/6$.

Complex Lottery

- A complex (compound) lottery is a $C = (L_1, L_2, \ldots L_n; q_1, q_2, ..q_n)$ where $q_1 + q_2 \ldots + q_n = 1$ and $q_i \geq 0$ for all i.
- Probability q_i represents the probability that lottery i is played.
- For example, if L_1 is a coin and L_2 is a die, then we can have $C = (L_1, L_2; 1/2, 1/2)$.
- What is the chance that heads will occur?
- What is the chance that the number 4 is rolled?

Reduced Lottery

- We can reduce a complex lottery into a simply lottery by just looking at the final probabilities.
- $L = q_1 * L_1 + q_2 * L_2 + \ldots . q_n * L_n$. (note the L's are vectors).
- In our coin/die example say that heads is the same outcome as 6 and tails is the same out come as 1.
- What is the reduced lottery (remember $C = (L_1, L_2; 1/2, 1/2)$)?

Continuity

- If preference over lotteries are complete, transitive and continuous, then as before there is a utility function that represents them, $L \succeq L' \Leftrightarrow U(L) \geq U(L')$.
- Example: Outcomes are $\{\$1000, \$10, \textit{death}\}$
- One may have lexicographic preferences for safety over money.
- This is not continuous.
- $W((\$1000, \$10, \textit{death}; 0, 1, 0))$ includes all the points $(\$1000, \$10, \textit{death}; 1 - x, 0, x)$ for $x > 0$, but not $x = 0$

Utility Functions and vNM

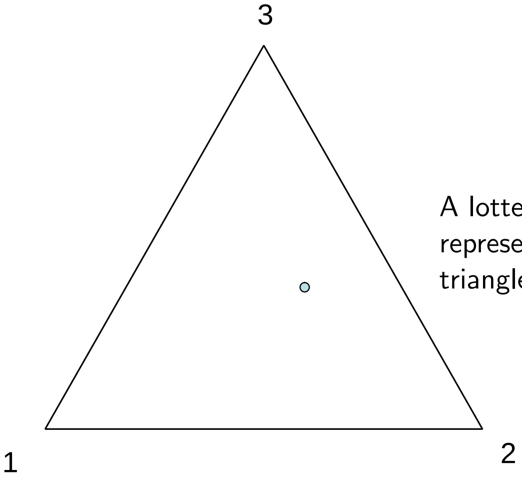
- A utility function has expected utility form (von Neumann-Morgenstern) if there is an assignment of numbers $(u_1, \ldots., u_n)$ to the n outcomes such that for every simply lottery $U(L) = p_1 u_1 + \ldots . p_n u_n$. (remember $L = (p_1, p_2, \ldots, p_n)$)
- Note that if $U(\cdot)$ is vNM, then $V(\cdot) = a_1 \cdot U(\cdot) + a_2$ where $a_1 > 0$ represents the same preferences and is also vNM. Prove it!
- Does $U(\cdot)$ and $V(\cdot) = U(\cdot)^2$ represent the same preferences if $U(\cdot) \geq 0$?
- Say outcome 1 is \$10 and outcome 2 is \$0. $L_1 = (1/2, 1/2)$, $L_2 = (1, 0), L_3 = (0, 1)$ and $U(\cdot)$ is *vNM*.
- Show that if U is vNM, we must have $U(L_1) = 0.5U(L_2) + 0.5U(L_3)$.
- Assume $U(L_3) = 0$. If $V(\cdot) = U(\cdot)^2$, then show that $V(L_1) = 0.25V(L_2)$.
- Can then $V(\cdot)$ be *vNM*? (Hint: show that if V is vNM, we must have $V(L_1) = 0.5V(L_2) + 0.5V(L_3)$.)

Utility Functions, Transformations and vNM

Theorem

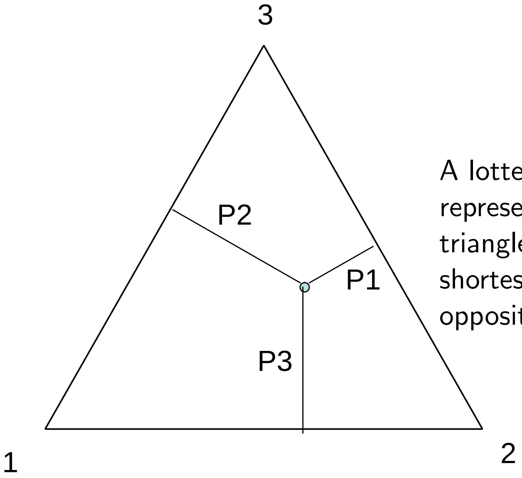
If $U()$ is vNM, then $V()$ is vNM and represents the same preferences iff $V(\cdot) = a_1 \cdot U(\cdot) + a_2$ where $a_1 > 0$.

Representing Lotteries with an Equilateral Triangle



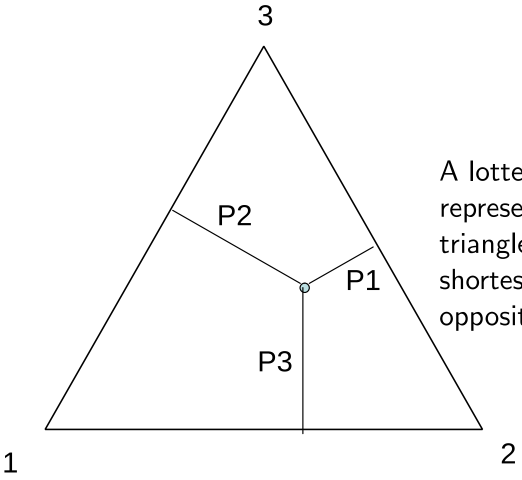
A lottery (p_1, p_2, p_3) is represented by a point in the triangle.

Representing Lotteries with an Equilateral Triangle



A lottery (p_1, p_2, p_3) is represented by a point in the triangle where p_i is the shortest distance to the side opposite the vertex.

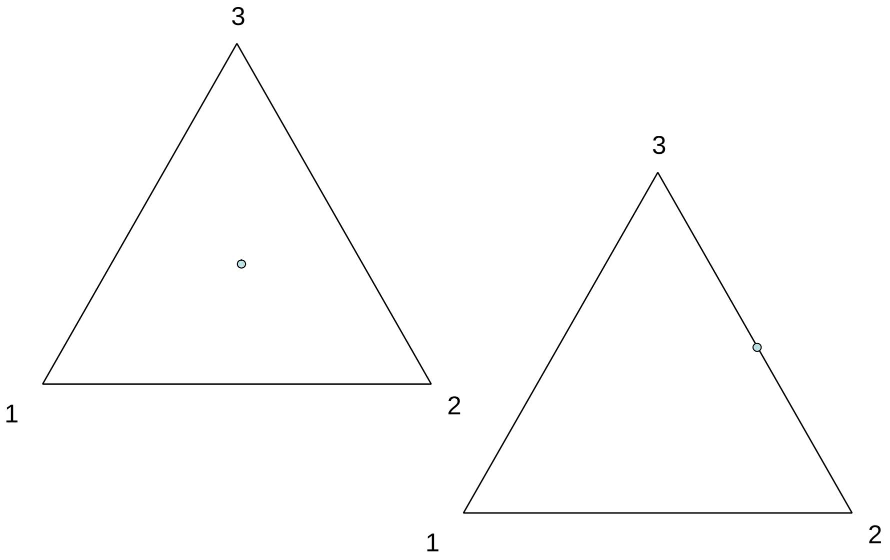
Representing Lotteries with an Equilateral Triangle



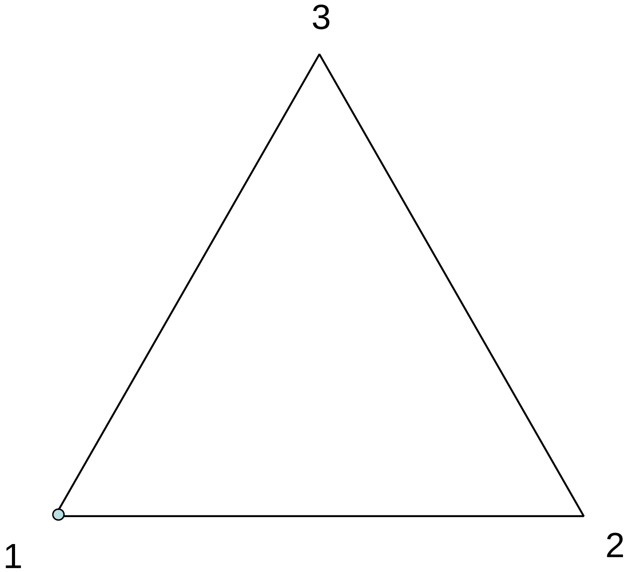
A lottery (p_1, p_2, p_3) is represented by a point in the triangle where p_i is the shortest distance to the side opposite the vertex.

- The sum of $p_1 + p_2 + p_3$ equals the height of the triangle.

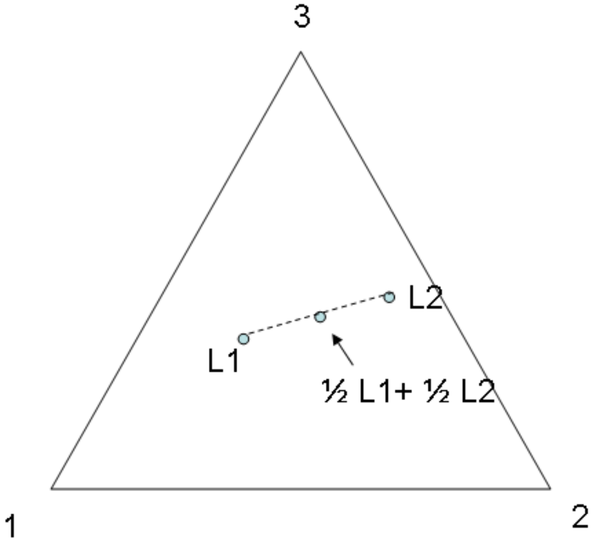
What do the following represent?



What does this represent?



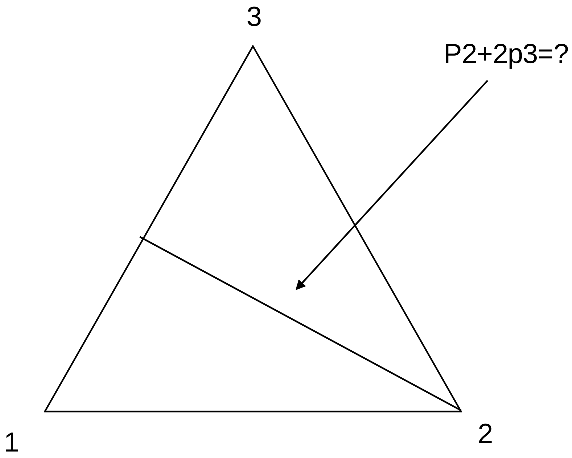
Reduced Lotteries in Triangles



Expected Utility: Indifference curves

- $u(3) = 3; u(2) = 2, u(1) = 1$.
- An indifference curve is a p_1, p_2, p_3 s.t. $p_1 + 2 \cdot p_2 + 3 \cdot p_3 = c$.
- We also must have $p_1 + p_2 + p_3 = 1 \Longrightarrow p_1 = 1 - p_2 - p_3$.
- Hence $p_2 + 2 \cdot p_3 = c$.
- This is a straight line.
- What are the end points for $p_2 + 2 \cdot p_3 = 1$?
  - When $p_2 = 0$ and when $p_3 = 0$.
- What are the end points for $p_2 + 2 \cdot p_3 = 1/2$?

Indifference Curves

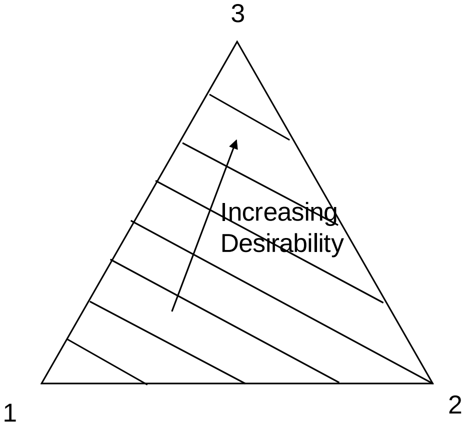


- Indifference Curves for EU are always parallel straight lines!
- Can you write the indifference curves just in terms of p_1 & p_2? p_1 & p_3?

Homework:

1. Draw the indifference curves for $u(3) = 2$, $u(2) = 3$, $u(1) = 1$, using an equilateral triangle to represent probabilities. Indicate the direction of increasing utility.
2. Draw the indifference curves for $u(3) = 2$, $u(2) = 2$, $u(1) = 1$, using an equilateral triangle to represent probabilities. Indicate the direction of increasing utility.

Indifference Curves



Can we have a utility function where the indifference curves are the same, but the desirability goes the opposite way?

Utility functions Complex

- Remember a complex lottery is $C = (L_1, L_2, \ldots L_n; q_1, q_2, ..q_n)$ and a decision maker cares only about the reduced lottery.
- A utility function is called linear over lotteries if $U(C) = q_1 U(L_1) + \ldots. + q_n U(L_n)$.

Theorem

A utility function is vNM iff it is linear.

Proof.

- Linear $\Longrightarrow$ vNM: Given U is linear we need to show $U(L) = p_1 u_1 + ...p_n u_n$. (hint: create a C whose reduced form lottery is L by choosing for L_1 s.t. $p_1 = 1$, etc.)
- vNM$\Longrightarrow$Linear: For simplicity, just show for $C = (L_1, L_2; 1/2, 1/2)$ where $L_1 = (1, 0; p, 1-p)$ and $L_2 = (1, 0; q, 1-q)$.
- Find $U(L_1)$, $U(L_2)$ and $U(\frac{1}{2}L_1 + \frac{1}{2}L_2)$. Use this to show that $U(C) = U(\frac{1}{2}L_1 + \frac{1}{2}L_2) = \frac{1}{2}U(L_1) + \frac{1}{2}U(L_2)$.

Independence Axiom

- A preference relation satisfies Independence if for all L, L', L'' and x we have $L \succeq L'$ if and only if $xL + (1-x)L'' \succeq xL' + (1-x)L''$.
- Show independence implies for all L, L' and x we have $L \succeq L'$ if and only if $xL + (1-x)L' \succeq L'$.
- **Homework:** Show independence (and transitivity) imply for all L, L', L'', L''' and x we have $L \succeq L'$ and $L'' \succeq L'''$, then $xL + (1-x)L'' \succeq xL' + (1-x)L'''$.
- Show independence implies for all L, L' and x we have $L \sim L'$ if and only if $xL + (1-x)L' \sim L'$.
- **Homework**: Show independence implies if for all L, L', L'' and x we have $L \sim L'$ if and only if $xL + (1-x)L'' \sim xL' + (1-x)L''$.

Expected Utility Theorem.

Theorem

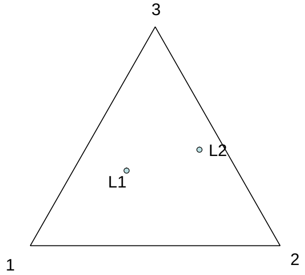
If a preference satisfies continuity and independence then it is represented by a vNM expected utility function.

Linear Utility in Triangles

Lemma

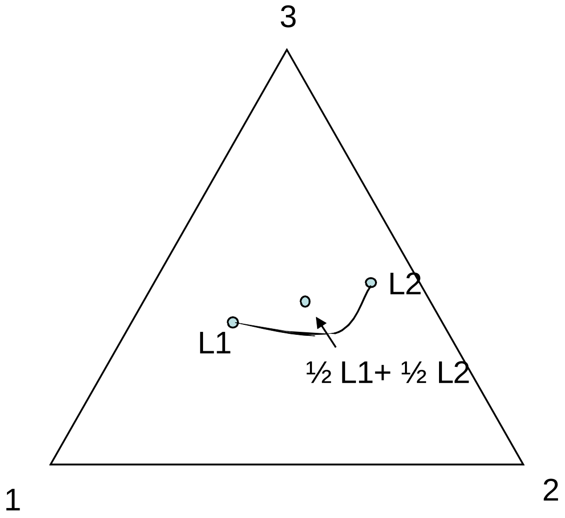
Linear utility implies indifference curves are straight lines.

- $U(L_1) = U(L_2) \Longrightarrow \alpha U(L_1) + (1-\alpha)U(L_2) = U(L_2)$.
- If utility is linear, then $C = (\alpha, 1-\alpha; L_1, L_2) \Longrightarrow$ $U(C) = \alpha U(L_1) + (1-\alpha)U(L_2) = U(L_2)$.
- Remember where is C located.



Violation of Independence

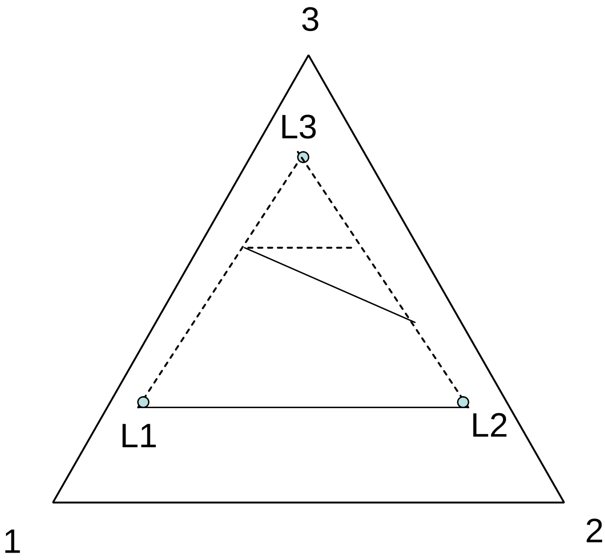
If utility is not linear (and hence not vNM), then independence will not hold.



Linear implies Independence

- $L \succeq L' \implies U(L) \geq U(L')$.
- $\implies xU(L) + (1-x)U(L'') \geq xU(L') + (1-x)U(L'')$.
- Why does linearity and this then imply $xL + (1-x)L'' \succeq xL' + (1-x)L''$?
- Show the other direction.

Must be Parallel lines as well.



Allais Paradox (1953)

- Choose between two gambles
  A. .33 chance of $27,500, .66 chance of $24,000 and .01 chance of nothing.
  B. $24,000 for sure.
- Another Choice between two gambles:
  C: .33 chance of $27,500 for sure and .67 chance of nothing.
  D: .34 chance of $24,000 for sure and .66 chance of nothing.

Violation

- Most choose B over A and C over D.
- If $A \prec B$, we have
- $.33 \cdot u(27,500) + .66 \cdot u(24,000) + .01 \cdot u(0) < u(24,000)$
- If $C \succ D$, we have??
- $.33 \cdot u(27,500) + .67 \cdot u(0) > .34 \cdot u(24,000) + .66 \cdot u(0)$
- Why is this a contradiction?
- Why did you make such choices?
- Overweight small probability events.
- .34 is similar to .33 but 27,500 is not similar to 24,000.

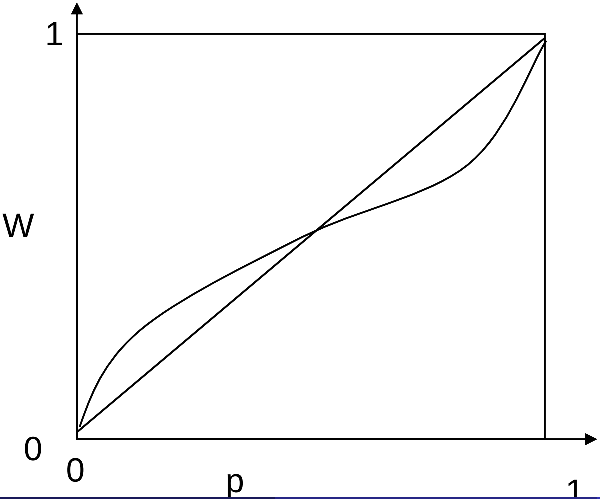
Allais Paradox 2

- Choose between two gambles
  A. 80% chance of $4,000
  B. $3,000 for sure.
- Another Choice between two gambles:
  C: 20% of $4000.
  D: 25% of $3000.
- Again readjusting probabilities or similarities seems to solve this paradox.

Incorporating Alias paradox into EU theory

(Prospect Theory, Kahneman and Tversky, 1979).

For *vNM* we had $U(L) = p_1 u_1 + \ldots . p_n u_n$.
One can adjust this to $U(L) = W(p_1) u_1 + \ldots . W(p_n) u_n$ where



Why does this work?

- A. 80% chance of \$4,000
  B. \$3,000 for sure.
- $W(.8)u(4000) < W(1)u(3000)$
- C: 20% of \$4000.
  D: 25% of \$3000.
- $W(.2)u(4000) > W(.25)u(3000)$
- What is a consistent u and W?
- We need $W(.2)/W(.25) > W(.8)$ for consistency.
- Say, $W(.8) = .7$, $W(.2) = .3$, $W(.25) = .35$, $u(x) = x$

Ellsberg Paradox (1961)

- An urn contains 300 coloured marbles: 100 are red and 200 are a mixture of blue and green.
- Choose one of the following:
  A: You receive \$1000 if the marble is red.
  B: You receive \$1000 if the marble is blue.
- Choose one of the following:
  C: You receive \$1000 if the marble is not red.
  D: You receive \$1000 if the marble is not blue.

Ellsberg Paradox

- Most choose A and C.
- Why is it a paradox?
- Expected utility implies that the utility of a gamble with p chance of winning is $p * u(1000) + (1 - p)u(0)$ and one would simply choose the lottery with the highest p.
- Let r, g, b denote the number of balls of the corresponding color. $r = 100$, $g + b = 200$.
- If one chooses A over B, then it implies $r > b$.
- If one choose C over D, then it implies $g + b > r + g$.
- Why did you make such a choice?
- This has been coined as "ambiguity aversion."

Ellsberg Paradox

- There are two urns: R and H with 100 balls in each. R contains 49 white balls and 51 black balls. H contains a mix of only white and black balls.
- I will pay you $100 if the ball is white. Which urn will you choose?
- I will pay you $100 if the ball is black. Which urn will you choose?

Can we explain Ellsberg with a W function?

- Let x equal probability of a white ball in urn H.
- We must have $W(.49) > W(x)$.
- Since W is increasing, this implies $.49 > x$
- We must also have $W(.51) > W(1 - x)$.
- This implies $.51 > 1 - x$.
- Together we must then have $1 > 1$ a contradiction!

Gilboa and Schmeider (1989)

- They have a model where people believe in Murphy's law.
- They have some prior possibility of proportions that are possible: $(b, g) = (100, 100)$ or $(50, 150)$ or $(150, 50)$.
- The decision maker figures whichever gives him the lowest expected payoff is the one it is.
- In the first choice the $(b, g) = ??$
  A: You receive \$1000 if the marble is red.
  B: You receive \$1000 if the marble is blue.
- In the second choice the $(b, g) = ??$
  C: You receive \$1000 if the marble is not red.
  D: You receive \$1000 if the marble is not blue.

Machina's paradox

- Three possible outcomes.
  1. Trip to Venice.
  2. Watching an excellent movie about Venice.
  3. Staying at home doing nothing.
- You may prefer $1 \succ 2 \succ 3$.
- You prefer a lottery of 99.9% trip 0.1% home to 99.9% trip 0.1% movie.
- Utility for a payoff depends upon the lottery: There is disappointment aversion.

Risk Aversion

- Say you have utility over money: $u(m)$.
- Risk aversion means it is concave: one prefers the certainty equivalence of a gamble:
  $U(x \cdot m_1 + (1-x) \cdot m_2) > xU(m_1) + (1-x)U(m_2)$.
- Arrow-Pratt Measure of risk aversion is $R_a = -\frac{u''}{u'}$. What is R_a for $u(x) = x$?
- What is R_a for $V = a \cdot U + b$?
- When $-\frac{u''}{u'} = c$, what is u?
- This is constant absolute risk aversion.
- Problem is that we may care about gambles compared to our wealth (adding a constant to wealth does not affect acceptance).

Relative Risk Aversion

- To correct for this, we can use CRRA (constant relative risk aversion).
- $Crra = -x \cdot u'' / u'$.
- What is CRRA of $V = a \cdot U + b$?
- What is the utility that satisfies $c = -x \cdot u'' / u'$?
- This is also written as $u(x) = \frac{x^{1-\rho}-1}{1-\rho}$.
- What is this when $\rho = 1$?

What Makes One Gamble Riskier than Another?

How risky is a gamble/investment (a lottery, L, where some payoff is negative)?

- Aumann-Serrano, 2008
  - At what level of risk aversion is CARA would one be indifferent to accepting the Lottery over zero: α such that $[e^{-\alpha \cdot w}|L] = 1$ (where w is the payoff of L).
  - Riskiness is $\frac{1}{\alpha}$ (higher number is riskier).
- Foster-Hart, 2009
  - Riskiness of gamble is amount of wealth one needs to avoid bankruptcy in the long run by accepting gambles below a preset risk level.
  - $E[Ln(1 + \frac{1}{R}w|L)] = 0$. Riskiness is R.
  - This is a similar utility to CRRA at $\rho = 1$.

Questions

1. A person has $u = \sqrt{w}$. His initial wealth is \$4. He has a lottery ticket that is worth \$12 with prob. $\frac{1}{2}$ and 0 with prob. $\frac{1}{2}$. What is his expected utility? What is the lowest price p that he will part with the ticket?
2. A person has $u(w) = e^{2 \cdot w}$. His initial wealth is \$0. He also has a lottery ticket that is worth $\$\ln 7$ with prob. $\frac{1}{2}$ and 0 with prob. $\frac{1}{2}$. What is his expected utility? What is the lowest price p that he will part with the ticket?
3. A consumer utility $u = \ln(w)$. He can bet on a coin with prob p of coming up heads. If he bets $\$x$, he will have $w + x$ for heads and $w - x$ for tails. What the optimal x as a function of p?