Optimal rewards in contests

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We study all-pay contests with effort-dependent rewards under incomplete information. The value to winning the contest for each contestant depends not only on his type but also on the effort-dependent reward chosen by the designer. We analyze which reward is optimal for the designer when his objective is either total effort or highest effort and when the value to a contestant for winning the contest is either multiplicatively separable or additively separable in reward and type. We find that when the value is multiplicatively separable, the optimal reward is always positive, while when the value is additively separable, it may also be negative. We also find that when the designer maximizes total effort and there is a sufficiently large number of contestants, the optimal reward decreases in the contestants' effort; however, when the designer maximizes the highest effort, the optimal reward may increase in the contestants' effort for any number of contestants. Finally, when we allow for the possibility of multiple rewards, we find that the designer's payoff depends only upon the expected value of the effort-dependent rewards and not on the number of rewards.

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1. Introduction

■ The Ansari X-prize was a ten-million-dollar competition created to jumpstart the space tourism industry by attracting the attention of the most talented entrepreneurs and rocket experts in the world.[1] This R&D contest is an example of a competition in which all contestants, including those that do not win any reward (prize), incur costs as a result of their efforts but only the winner gets the reward. Such winner-take-all contests take many other forms: only the first firm to invent gets a patent, the hedge fund that finds the arbitrage opportunity the quickest gets the profits, the first runner to cross the finish line wins a marathon, and only the best worker gets the promotion.

In many situations, there is a relationship between the efforts made in the contest and the size of the reward collected by the winning contestant. In the X-prize as with patent races, the winning firm choosing a larger effort leads to an earlier innovation time. This in present value terms leads to a larger reward. A hedge fund not only faces competition from other hedge funds, but from market forces eliminating opportunities. Earlier detection can lead to larger profits. In the marathon, harder training can lead to a quicker winning time. This can result in a larger reward (such as if a course or world record is broken). In work promotions, greater effort can result in a larger raise to the winner.

Also in many cases, the sponsor has at least a limited control over the design used: the government can determine scope and length of patents, the SEC can regulate hedge funds, the organizers of the marathon can set the size of the prize, and the company can set rules with a promotion contest.

[1] The cash prize was awarded to the team headed by Burt Rutan and Paul Allen who were the first to privately finance, build and launch a spaceship able to (a) carry three people to 100 kilometers (62.5 miles), (b) return safely to earth, (c) repeat the launch with the same ship within two weeks. The X-prize was inspired by the early aviation prizes of the 20th century, primarily the spectacular trans-Atlantic flight of Charles Lindbergh in the Spirit of St. Louis which captured the $25,000 Orteig prize in 1927. See www.xprizefoundation.com for a description of the contest, winners, and inspiration from the Orteig prize.

There has been initial research in contests with the aforementioned effort–dependent rewards. Kaplan et al. (2002, 2003) show that in contests under incomplete and complete information, for particular effort-dependent rewards there are substantial qualitative changes to the behavior of the contestants compared with constant reward contests. In addition, there is a growing literature of contest design. Baye et al. (1993), Taylor (1995), Fullerton and McAfee (1999) found advantages to limiting the number of contestants. Che and Gale (1998), Gavious et al. (2003) and Kaplan and Wettstein (2006) all analyze the profitability of bid caps. Barut and Kovenock (1998) and Moldovanu and Sela (2001) study fixed-reward contests, where the designer can determine the number of prizes having positive value and the distribution of the fixed total reward among the different prizes.

The contribution of this paper is that we combine contest design with effort-dependent rewards, that is, we allow the designer of the contest to choose how the rewards depend upon efforts. We study this design problem under incomplete information about the contestants' types. We also use a two-by-two framework: there are two possible objectives for designer, maximize either the total expected effort or the expected highest effort of the contestants (minus reward paid), and there are two possible preferences of the contestants, the value to winning is multiplicatively separable in reward and type and the value to winning is additively separable in reward and type.

This framework is useful since we find results about the equilibrium effort and optimal reward among our environments that depend significantly upon the common objective of the designer or preference of the contestant. If the contest designer wishes to maximize the expected total effort, for sufficiently large number of contestants, the optimal reward function decreases in effort, that is, a larger effort decreases the size of the reward gained by winning. On the other hand, if the contest designer wishes to maximize the expected highest effort, for any number of contestants, the optimal reward function may increase in effort. While the possibility of decreasing rewards may seem surprising, it occurs since when the designer maximizes total effort, he induces an effort that depends only on type and not on the number of contestants. However, given this, as the number of contestants increases, the probability of winning for low effort decreases at a faster rate than that for higher effort. Thus, the reward of winning at those efforts must be increasing at a faster rate in order to maintain the same expected payoff for a particular effort. Eventually, this must cause the reward to be decreasing.

An additional finding is that with multiplicatively separable values, the optimal reward is always positive, while with additively separable values, it may also be negative. Furthermore, with multiplicatively separable values, the optimal reward function induces all contestants (even those with the lowest types) to choose to participate in the contest. On the other hand, the optimal reward function with additively separable values may limit the number of contestants that choose to participate in the contest. This can be interpreted as the optimal reward structure serving the role of entry fees or alternatively reserve prices in the standard contests (or auctions). While we study the designer choosing the effort-dependent reward structure in a contest, the design that we find is optimal among all possible contests and when the designer's objective is total expected effort, it is optimal over all mechanisms.[2]

We expand our analysis to allow the designer control of the number of effort-dependent rewards. Contrary to Moldovanu and Sela (2001), we find that it is not the number of prizes to which rewards are distributed that matters, but it is only the expected value of the reward. For instance, we present an example where an optimal design is to give an effort-dependent reward to the loser rather than the winner of a two-player contest (in such a design, as long as one isn't the best, putting forth higher effort increases reward, emulating some scenarios in life). This surprising result further demonstrates that an effort-dependent reward can be an efficient tool for the contest designer but

[2]See the end of Section 4 for further discussion.

its structure as well as its effects are sometimes unusual.

The paper proceeds as follows. In Section 2 we present the model. In Sections 3 and 4, we analyze the optimal reward when the contest designer wishes to maximize the expected total effort and the expected highest effort, respectively. In Section 5, we revisit the question of multi-reward contests and in Section 6 we conclude. The Appendix contains the proofs.

2. The model

■ Consider $n-$player, all-pay contests with effort-dependent rewards. Each contestant's type $\theta_i, i \in \{1, ..., n\}$, is independently drawn from the interval $[\underline{\theta}, \overline{\theta}]$, where $0 \leq \underline{\theta} < \overline{\theta}$, according to the same cumulative distribution function F. We assume that there are no atoms; there is positive density, $f(\theta) > 0$; and that the hazard rate, $f(\theta)/(1 - F(\theta))$, is increasing in θ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. While F is common knowledge, each contestant is privately informed about his own type. Each contestant i produces an observable effort x_i and, by doing so, incurs a nonobservable disutility (or cost) denoted by $c(\theta_i, x_i)$, where $c : [\underline{\theta}, \overline{\theta}] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing in x (a higher effort is more costly), strictly decreasing in θ (higher types have lower costs), and twice continuously differentiable. We also assume $c(\theta, 0) = 0$ (no effort is costless), $c_x(\theta, 0) = 0$ (a small effort is near costless) and $\lim_{x \rightarrow \infty} c_x(\theta, x) > 1$ (at some point the marginal cost of effort is strictly larger than one).[3] There are additional conditions assumed on c, which are sufficient to guarantee a monotonic solution to our problem: $c_{xx} > 0$, $c_{x\theta} < 0$ for all $x > 0$ and $c_{\theta x}(\theta, 0) = 0$, $c_{x^2\theta} \leq 0$, $c_{x\theta^2} \geq 0$.[4] That is, there is a diminishing marginal product ($c_{xx} > 0$),[5] higher types have less of a diminishing marginal product ($c_{x^2\theta} \leq 0$), and higher types are less averse to increases in effort ($c_{x\theta} < 0$).

[3]In the language of an all-pay auction, x is the bid and $c(\theta, x)$ is the cost of bidding. Since the environment is one of a contest, we find it semantically simpler to call x the effort (that is, the result of an exertion) and $c(\theta, x)$ the cost of effort (as in, the cost of the exertion itself).

[4]This sufficiency we will verify later.

[5]More precisely, this condition implies that the production function associated with this cost function exhibits diminishing marginal product.

The contestant i that chooses the highest effort x_i wins a reward $R(x_i)$, where $R : \mathbb{R}_+ \to \mathbb{R}$ is continuous,[6] and values winning in the contest according to the function $V(\theta_i, R(x_i))$, where $V : [\underline{\theta}, \overline{\theta}] \times \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function with $V_\theta \geq 0$.[7] This value to winning can be different from the reward paid by the designer, particularly, since it may include any psychological gains or additional economic benefits to winning.[8]

We will consider two specific forms of this value function: multiplicatively separable (**mult**), $V(\theta, R(x)) = \theta \cdot R(x)$, and additively separable (**add**), $V(\theta, R(x)) = \theta + R(x)$. The **add** value function has the type-dependent component to winning independent of the size of the reward. This can be the case where there is prestige or status associated with winning (or signal from winning) that is not derived from the monetary value of the reward. This may be the case with many sports contests. The **mult** value function captures the case where the benefit from the monetary value of the reward is related to type. Such a relationship may exist in technology contests, where a larger reward attracts more media attention and the ability to capitalize on this extra attention is increasing in type. It could also simply be that the psychological value to winning depends upon the size of the reward (or trophy) of winning.

We also consider two possible objectives for the designer: the designer maximizes the expected value of total effort $E[\sum_{i=1}^{n} x_i]$ minus the expected reward he must pay out (**total**), and the designer maximizes the expected value of the highest effort $E[\max\{x_i\}]$ minus the expected reward he must pay out (**highest**). The **total** objective has the designer valuing efforts by all contestants; this mirrors situations such as promotion contests. The **highest** object has the designer not

[6]The assumption of continuity of the reward function is made for simplicity of analysis and our results would still hold if it is relaxed. Moreover, it can be replaced by the realistic assumption that there is a small amount of noise in determining the winner.

[7]The assumption that $V_\theta \geq 0$ implies that the value to winning ***may*** depend upon type and if so contestants with lower costs also value winning more. The other case where $V_\theta < 0$ can be treated as well, but then the equilibrium does not necessarily exist given the assumption about the cost function. A step further would have a second signal for the value function with a joint distribution over both signals. This would complicate analysis considerably.

[8]For instance, the winners of the X-prize received additional reward from the 10 million prize in the form of a contract from Virgin to form Virgin Galactic (see www.virgingalactic.com).

benefiting from the efforts of the losers; this fits scenarios such as technology contests or patent races. Our two-by-two framework yields four environments, which we denote by **total-mult, total-add, highest-mult**, and **highest-add**.

The timing of decisions is the designer chooses the reward function and afterwards the contestants see their individual types and choose efforts. Each contestant i chooses his effort x_i in order to maximize his expected utility, given his type, the other contestants' actions and the form of the reward function.

□ **Equilibrium.** Consider the equilibrium in the contest that results after the designer sets the reward function. Denote the equilibrium expected utility (profit) of a contestant of type θ by $\pi(\theta)$. In a Bayesian equilibrium, the effort function $x(\theta)$ chosen by each contestant maximizes his expected utility given the effort functions chosen by the other contestants. Hence, for each θ, a symmetric equilibrium effort function $x(\theta)$ (assumed to be strictly increasing and continuous with inverse $\theta(x)$) solves the following profit maximization problem:[9]

$$\pi(\theta) \equiv \max_x F(\theta(x))^{n-1} \cdot V(\theta, R(x)) - c(\theta, x). \tag{1}$$

Proposition 1. Any equilibrium strategy $x(\theta)$ is given by the implicit function

$$F(\theta)^{n-1}V(\theta, R(x(\theta))) - c(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1}V_\theta(\tilde{\theta}, R(x(\tilde{\theta}))) - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta}. \tag{2}$$

Proof. See the Appendix.

We derived (2) by using the Envelope Theorem to find the RHS of the equation and setting it equal to the contestant's problem (1) at the solution. Equation (2) generalizes the solution for a range of auction mechanisms. For instance, if $c(\theta, x) = 0$ for all x, θ and $V(\theta, R(x)) = \theta - x$, we

[9] These properties are actually induced by the designer's choice of $R(x)$ as we show later in the Appendix 7.

have a first-price auction. From (2), we have

$$F(\theta)^{n-1}(\theta - x(\theta)) = \int_{\underline{\theta}}^{\theta} F(\tilde{\theta})^{n-1} d\tilde{\theta}$$

which is consistent with the known solution of a first-price auction.

If $c(\theta, x) = x$ for all θ and $V(\theta, R(x)) = \theta$ for all x, we have a standard all-pay auction. Again from (2), we have a solution consistent with known results, namely,

$$F(\theta)^{n-1}\theta - x(\theta) = \int_{\underline{\theta}}^{\theta} F(\tilde{\theta})^{n-1} d\tilde{\theta}.$$

In order to solve the designer's problem, we must first choose the designer's objective. Then, we must use the fact that a designer's choice of reward function influences the equilibrium behavior of the players in the subsequent contest. This allows us to use the solution of the contest equilibrium to replace the expected rewards the designer must pay out $E[R(x(\theta))|\theta$ is the highest type] with a function of $x(\theta)$. This converts the designer's problem from finding the best reward function to one of finding the optimal induced effort $x(\theta)$. Finally, once we have the optimal induced effort, we can find the reward function $R(x)$ that induces that effort.

3. Maximization of the total effort

■ In this section, we analyze the case in which the designer wishes to maximize the expected value of total effort net of the expected reward he must pay out:

$$n\int_{\underline{\theta}}^{\overline{\theta}} x(\theta) dF - \int_{\underline{\theta}}^{\overline{\theta}} R(x(\theta)) dF^n. \tag{3}$$

The left term of (3) is the expected total effort exerted by the contestants, while the right term is the designer's expected payment to the contestant with the highest effort. Due to the difficulty of solving the contest induced by a designer choosing a reward function, we can only do so, once we make further assumptions about the value function.

□ **Multiplicatively separable case.** We consider first the **total-mult** environment where the value function is multiplicatively separable, $V(\theta, R(x)) = \theta \cdot R(x)$. The equilibrium effort $x(\theta)$ is the solution of the following maximization problem:

$$\arg\max_x F(\theta(x))^{n-1} \cdot \theta \cdot R(x) - c(\theta, x).$$

Equivalently, this equilibrium effort $x(\theta)$ is also the solution of the problem

$$\arg\max_x F(\theta(x))^{n-1} \cdot R(x) - \hat{c}(\theta, x) \tag{4}$$

where $\hat{c}(\theta, x) = c(\theta, x)/\theta$ and any solution to this problem is also a solution to the original problem. Hence, we can instead solve the latter problem. This we call the type-independent case where the value function does not depend on type, namely,

$$V(\theta, R(x)) = R(x), \tag{5}$$

and the cost of effort is $\hat{c}(\theta, x) = c(\theta, x)/\theta$.[10] This new problem is easier to solve and leads to the following proposition.

[10] In this case, we only need for c/θ to satisfy our sufficient conditions. Note that if the sufficient conditions hold for c, they will also hold for c/θ when $\theta > 0$.

Proposition 2. In the **total-mult** environment, the optimal reward is given by

$$R(x) = \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x}) d\theta(\tilde{x}) \right) / F(\theta(x))^{n-1} \tag{6}$$

where $\hat{c}(\theta, x) = c(\theta, x)/\theta$ and $\theta(x)$ is the inverse of the equilibrium effort $x(\theta)$ which is given by

$$1 + \hat{c}_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = \hat{c}_x(\theta, x(\theta)). \tag{7}$$

Proof. See the Appendix.

By using Proposition 2, we derive some properties of the contestants' equilibrium efforts and the optimal reward in this environment.

Proposition 3. In the **total-mult** environment,

1. The equilibrium effort is independent of the number of contestants n.
2. All contestants choose to participate in the contest.
3. The optimal reward is always positive.
4. For large enough n, the optimal reward is decreasing.

Proof. Equations (6) and (7) imply points 1, 2 and 3. The proof of point 4 is proved in the Appendix.

The main result of Proposition 3 is that for a sufficiently large number of contestants, the optimal reward function decreases in effort, that is, a larger effort decreases the size of the reward gained by winning. In the following example, we see that in a contest with as few as six contestants, the reward function can be decreasing.

Example 1. In the **total-mult** environment, the cost function is $c(\theta, x) = x^2$ and the distribution of the contestants' types F is uniform on $[0, 1]$.

From this specification, $\hat{c}(x, \theta) = x^2/\theta$ and we can rewrite (7) as

$$1 - (\frac{2x(\theta)}{\theta^2})(1 - \theta) = 2x(\theta)/\theta.$$

Thus, the equilibrium effort is

$$x(\theta) = \frac{\theta^2}{2}.$$

Consistent with Proposition 3, the optimal effort does not depend upon n. The inverse of the equilibrium effort is $\theta(x) = \sqrt{2x}$. By (6) the optimal reward is

$$R(x) = \frac{(2x)^{2-n/2}}{3}.$$

For two contestants the optimal reward is $R(x) = 2x/3$. For four contestants, we have an effort-independent optimal reward $R(x) = 1/3$. For six contestants, we have already a decreasing reward function $R(x) = 1/(6x)$.

Perplexing questions arise from Proposition 3 and this example. First, why would the equilibrium effort be independent of the number of contestants: given the all-pay nature of the contest wouldn't adding players dissipate individual effort as with a standard all-pay auction? Second, why would a designer want to reward winning contestants less the **harder** they work: would this not dampen effort? Third, even if it is optimal, can the reward ever realistically be decreasing? Finally, why isn't it obvious that all the contestants participate?

The answer to the first question is that the designer values all efforts whether or not they are the highest. This value for individual effort is not based upon the number of contestants. Furthermore, the cost of maintaining an individual's effort depends upon the expected payment to that contestant and does not increase with the number of contestants. Thus, in our model, if the designer has the tools, he would want to induce efforts independent of n by increasing the reward $R(x)$ as the number of contestant increases in such a way to keep the equilibrium efforts constant. With effort-dependent rewards that can be chosen after learning n, the designer has such tools. If, as in the standard all-pay auction, the designer had to keep the reward for winning constant, $R(x) = R$, or could not vary the reward with the number of contestants, the equilibrium efforts would indeed depend upon n.[11]

[11] We see this for a constant reward by looking at the environment of Example 1 (**total-mult** environment, $c(\theta, x) = x^2$ and $F(\theta) = \theta$), for a constant reward R, the equilibrium effort is then given by $x(\theta)^2 = R((n-1)/n)\theta^n$ and does indeed depend upon n. In fact, no matter what reward R the designer chooses, the effort will still depend upon n since the θ^n term will remain as long as $R > 0$.

If the designer could not increase the reward with n, the efforts would disspate. Indeed, Kaplan et al. (2002) solve the all-pay auction for a fixed variable reward $R(x)$, that is, one not chosen by a designer, and find the equilibrium efforts do depend upon n.

The answer to the second question, why would a designer want to reward winning contestants less the harder they work, is that the contestant cares about the expected payoff for winning (value of reward for winning times the chance of winning) rather than simply the reward for winning. As we see from the example, the expected payoff is

$$R(x)\theta(x) \cdot \theta(x)^{n-1} = (2x)^{2-n/2}/3 \cdot \sqrt{2x} \cdot (\sqrt{2x})^{n-1} = (2x)^2/3.$$

This expected payoff is always increasing. From the first question, the designer induces efforts independent of n. However, given this as the number of contestants increase, the probability of winning for low effort decreases at a faster rate than that for higher effort. Thus, the reward for winning at those efforts must be increased at a faster rate. Eventually, this must cause the reward to be decreasing.

This leads us to the third question. Can reward ever realistically be decreasing in effort? This is not as unusual as one may first suspect. For example, take contests where the first to achieve the task gets the reward (such as with innovation contests). It is quite possible that the reward is increasing over time. For instance, with innovation contests similar to the X-prize, the organizers may continue to raise funds by finding sponsors. A mathematician proving the Riemann Hypothesis in 2008 will receive a million dollars from the Clay Mathematics Institute for solving one of the Millennium Problems. However, it is quite possible that the institute will get an additional sponsor increasing the prize money to solving the hypothesis in 2009 to say two million dollars. Hence, the mathematican would be compensated more (in both nominal and present value terms) if he made the discovery later (and given that others do not make the discovery meanwhile). Why is this a decreasing reward? Simply that innovating earlier requires a greater effort:[12] time and effort go in

[12]Introducing noise into the production technology for innovation will allow some early innovators to simply be "lucky" and save effort with an early innovation, but those *choosing* an earlier expected innovation time will expend more effort on average for this choice.

opposite directions.

Finally, we explain the answer to the last question, why isn't it obvious that all the contestants participate? In optimal auctions for standard private-value environments (which includes all-pay auctions) the optimal design is to set a minimum bid. This causes some contestants not to enter (bid zero). This is because the designer tries to include only those contestants with positive virtual valuations (marginal revenue). In standard auctions, the virtual valuation of contestant with type θ is $\theta - \frac{1-F(\theta)}{f(\theta)}$. When the types are uniformly distributed on $[0, 1]$, this equals $2\theta - 1$. As one can easily see, this can be negative. In our case, the designer can obtain (only) positive virtual rent from all the contestants. We will go into further detail with this point in the additively separable case.

To illustrate the answer to the first question further, the independence of the equilibrium effort in the number of contestants becomes clear if we notice that the optimal reward $R(x)$ in the multiplicatively separable environment when the designer values expected total effort is comparable to the optimal wage contract in a Principal-Agent (PA) model (with n agents and hidden information about ability) where the principal offers each agent a wage $w(x)$ that depends upon output x (which can be sold at a price of one). In the PA model the agent's maximization problem is

$$\max_{x} w(x) - \hat{c}(\theta, x). \tag{8}$$

The principal's expected payoff given the solution $x(\theta)$ of the agent's problem is the expected output that he receives (the price of which is one) minus the expected wage that he must pay:

$$n \int_{\underline{\theta}}^{\overline{\theta}} [x(\theta) - w(x(\theta))] dF. \tag{9}$$

The substitution of $w(x) \equiv R(x) \cdot F(\theta(x))^{n-1}$ into (8) and (9) yields the problems of (4) and (3), respectively.

From the above analysis, we see that there is a competitive solution to the classical PA model. Instead of offering every agent a wage that depends on his output in the PA model, the principal can offer the agents a contest where the agent with the highest effort wins a reward that is dependent upon his output.

While we haven't performed a general mechanism design analysis, by this equivalence between our contest in environment **total-mult** and the PA model, a contest with the optimal effort-dependent reward is an optimal mechanism.

□ **Additively separable case.** We now analyze the **total-add** environment (where the value function is additively separable, $V(\theta, R(x)) = \theta + R(x)$.

Proposition 4. In the **total-add** environment, the optimal reward is given by

$$R(x) = \left(c(\theta(x), x) + \int_{\theta^*}^{\theta(x)} [F(\theta)^{n-1} - c_\theta(\theta, x(\theta))] d\theta \right) / F(\theta(x))^{n-1} - \theta(x) \tag{10}$$

where $\theta(x)$ is the inverse of the equilibrium effort $x(\theta)$ that is determined by

$$1 + c_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = c_x(\theta, x(\theta)) \tag{11}$$

and the cutoff θ^* is the θ that maximizes the designer's profit from the set $\{\theta \in [\underline{\theta}, \overline{\theta}] : x(\theta) - c(\theta, x(\theta)) + \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta)) \frac{1-F(\theta)}{f(\theta)} = 0\}$ (if the set is empty $\theta^* = \underline{\theta}$).

Proof. See the Appendix.

The equilibrium here has both qualitative similarities and qualitative differences to the multiplicative case (Propositions 2 and 3). We see this from the following proposition.

Proposition 5. In the **total-add** environment,

1. The equilibrium effort is independent of the number of contestants.

2. Some contestants may not choose to participate in the contest.
3. The optimal reward is not always positive.
4. For large enough n, the optimal reward is decreasing.

Proof. Equations (10) and (11) imply points 1 and 2. Point 3 is established by the following example and point 4 is proved in the Appendix.

The results of this proposition are made apparent with the following example.

Example 2. In the **total-add** environment, the cost function is $c(\theta, x) = x^2/\theta$ and the distribution of the contestants' types F is uniform on $[0, 1]$.

By (11) the optimal effort is

$$x(\theta) = \frac{\theta^2}{2}.$$

Notice that here the optimal effort function does not depend upon n. The cutoff θ^*, however, does depend upon n as we see by the equation that determines the cutoff:

$$\frac{\theta^{*2}}{4} + 2\theta^{*n} - \theta^{*n-1} = 0.$$

For $n = 2$, this has two solutions: $\theta^* = 0$ and $4/9$. The designer's profit (from equation (A5) in the Appendix 7) is given by

$$2\int_{\theta^*}^{1}\left[\frac{\theta^2}{2} - \frac{\theta^3}{4} + (2\theta - 1)\,\theta - \frac{\theta^2}{4}\,(1-\theta)\right] d\theta = \frac{1}{2} + \theta^{*2}(1 - \frac{3}{2}\theta^*).$$

This implies that the designer will maximize profits by choosing a cutoff of $\theta^* = 4/9$. By (10) this yields an optimal reward function of

$$R(x) = \frac{2x}{3} - \frac{(2x)^{1/2}}{2} - \frac{8 \cdot 29}{3^7(2x)^{1/2}}.$$

Notice that the expected payment is $-1145/4374$ which is negative.

From Proposition 5, we see that in the additively separable case as in the multiplicatively separable case, if the contest designer wishes to maximize the expected total effort, the equilibrium effort is independent of the number of contestants. Moreover, while not explicit in the example, for a sufficiently large number of contestants, the optimal reward function in both environments decreases in effort. The reasons for both of these results is similar to those for the **total-mult** environment.

On the other hand, while in the multiplicatively separable case all contestants participate in the contest, in the additively separable case some of them may choose to not participate in the contest, that is, the designer is able to indirectly eliminate contestants by inducing them not to participate by setting the reward function such that a contestant with a low type will do better by not participating. Furthermore, reward can be negative.

From this, one may again ponder about the following two questions. First, why is it now optimal to reduce participation? Second, why would anyone choose an effort where the reward to winning is negative?

Now, in order to see why it is optimal to eliminate some contestants in the additively separable case we examine the designer's payoff, which is given by

$$n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - c(\theta, x(\theta)) + \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} \right] dF.$$

The reward function in this case can limit the participation of players for which the expression within the integral is negative. To illustrate this possibility, take for example the case where the cost function does not depend on the contestant type, i.e., $c(\theta, x) = x$. This converts the environment into the standard auction environment where revenue equivalence holds. In accordance, the designer's payoff reduces to $\int_{\underline{\theta}}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta)}{f(\theta)} \right) dF^n$. (This is possible since (11) holds for any possible $x(\theta)$.) Hence, the designer's payoff depends only upon which types are included. Since, as with the standard auction, $\left(\theta - \frac{1-F(\theta)}{f(\theta)} \right)$ can be negative, some types may be excluded. We also note that in the multiplicatively separable environment participation is always optimal since the expression inside the integral there,

$$x(\theta) - \hat{c}(\theta, x(\theta)) + \hat{c}_\theta(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)},$$

is always positive.

The answer to the second question, why would anyone choose an effort level where the reward is negative is simple. The value to winning (the value to the object plus reward), $\theta + R(x(\theta))$, is always positive. For instance, in our example,

$$\theta + R(x(\theta)) = \theta + R(\theta^2/2) = \theta + \theta^2/3 - \theta/2 - (8 \cdot 29/3^7)\theta.$$

At the cutoff, $\theta^* = 4/9$, we have $\theta + R(x(\theta^*) = 4/81$. Since in our example $\theta + R(x(\theta))$ is clearly an increasing function, it is always positive.

Similar to before, under our assumptions in environment **total-add**, a contest with the optimal effort-dependent reward is an optimal mechanism in many contexts. However, there are some contexts where the designer could benefit from additional flexibility. For instance, if we assume that the mechanism designer can deny a winner of the contest the benefit of θ while still paying $R(x)$, then our solution would not be the optimal mechanism. This may be realistic in the context of promotion contests. In this case, a designer may be able to give a raise without giving a promotion. There $R(x)$ would represent the raise and θ would represent the status and other benefits gained from the titular promotion. Here, the optimal mechanism will involve minimum effort such that if the winner achieves this, he will get he promotion plus $R(x)$ and if the minimum is not achieved, he will receive only $R(x)$. We note that the minimum bid will be set such that type that precisely bids it is θ^*, where $\theta^* - \frac{1-F(\theta^*)}{f(\theta^*)} = 0$. In addition, in contrast to **total-add,** participation will again be always optimal as in **total-mult**. Returning to Example 2, when the θ can be given separately, the optimal effort is still $x(\theta) = \frac{\theta^2}{2}$. However, the minimum bid is set to $1/8$ (with $\theta^* = 1/2$) and optimal reward is now given by

$$R(x)=\begin{cases}\frac{c(\theta(x),x)-\int_0^{\theta(x)}c_\theta(\theta,x(\theta))d\theta}{F(\theta(x))^{n-1}}=\frac{(2x)^{2-n/2}}{3} & \text{if } x<\frac{1}{8},\\ \frac{c(\theta(x),x)-\int_0^{\theta(x)}c_\theta(\theta,x(\theta))d\theta+\int_{1/2}^{\theta(x)}F(\theta)^{n-1}d\theta}{F(\theta(x))^{n-1}}-\theta(x)=\frac{(2x)^{2-n/2}}{3}+\frac{\sqrt{2x}}{n}-\frac{(2x)^{(1-n)/2}}{2^n n} & \text{if } x\geq\frac{1}{8}.\end{cases}$$

In the following section, the designer wishes to maximize the highest effort and in this case our model would not necessarily be optimal even in the multiplicatively separable case.[13]

4. Maximization of the highest effort

■ Assume now that the designer cares about the expected value of the highest effort instead of expected total effort. In this case, his expected payoff is given by

$$\int_{\underline{\theta}}^{\overline{\theta}}[x(\theta)-R(x(\theta))]dF^n. \tag{12}$$

□ **Multiplicatively separable case.** When the value function is multiplicatively separable, the optimal reward is given in the following proposition.

Proposition 6. In the **highest-mult** environment, the optimal reward is given by

$$R(x)=\left(\hat{c}(\theta(x),x)-\int_0^x\hat{c}_\theta(\theta(\tilde{x}),\tilde{x})d\theta(\tilde{x})\right)/F(\theta(x))^{n-1} \tag{13}$$

where $\hat{c}(\theta,x)=c(\theta,x)/\theta$ and $\theta(x)$ is the inverse of the equilibrium effort $x(\theta)$ which is determined by

$$F(\theta)^{n-1}+\hat{c}_{\theta x}(\theta,x(\theta))\frac{1-F(\theta)}{f(\theta)}=\hat{c}_x(\theta,x(\theta)). \tag{14}$$

Proof. See the Appendix.

As a function of the equilibrium effort, the reward formula where the designer maximizes the

[13]Even if the designer is restricted to receiving messages only through effort (the mechanism must be all-pay), the designer may want to employ an asymmetric mechanism that eliminates some contestants independent of type.

highest effort (13) is the same as in the case where the designer maximizes the total effort (6). However, since the equilibrium efforts are not identical in both cases (equations (7) and (14) are not equal) we obtain that the optimal rewards are different.

It is important to notice that in the case when the designer values the expected highest effort, the optimal reward $R(x)$ in the multiplicatively separable environment is not comparable to any variable in the classical hidden-information Principal-Agent (PA) model. Indeed, in this case the properties of the contestants' equilibrium efforts and the optimal reward are completely different than their properties when the designer maximizes the expected value of total effort. In particular, it is shown that in the case when the designer maximizes the expected value of total effort, the optimal reward function decreases in effort if the number of players is sufficiently large. In this case where the designer values the highest effort, the optimal reward may be increasing for any number of players.[14]

Proposition 7. In the **highest-mult** environment,

1. The equilibrium effort depends on the number of contestants.
2. All contestants choose to participate in the contest.
3. The optimal reward is always positive.
4. The optimal reward may be increasing for any number of contestants.

Proof. Equations (13) and (14) imply points 1-3. Point 4 is illustrated in the following example.

Example 3. In the **highest-mult** environment, the cost function is $c(\theta, x) = x^2$ and the distribution of the contestants' types F is uniform on $[0, 1]$.

Thus $\hat{c}(\theta, x) = x^2/\theta$. From this specification, we can rewrite (14) as

$$\theta^{n-1} - (\frac{2x(\theta)}{\theta^2})(1-\theta) = \frac{2x(\theta)}{\theta}.$$

This implies the equilibrium effort

$$x(\theta) = \frac{\theta^{n+1}}{2}.$$

[14] In addition, in the multiplicatively separable environment, the equilibrium effort function when the designer values the expected highest effort is point-wise smaller than the equilibrium effort when the designer values the expected total effort. Thus, the expected value of the highest effort is smaller when the designer values the highest effort.

The inverse of the equilibrium effort is $\theta(x) = (2x)^{1/(n+1)}$. By (13) the optimal reward is then

$$R(x) = \frac{2n+2}{4(2n+1)} (2x)^{\frac{n+2}{n+1}} .$$

It can be verified that for large n this reward approaches $R(x) = x/2$.

Notice that the expected highest effort in this case is $n/(4n+2)$ while the expected highest effort in the case where the designer maximizes the total effort is always larger and equal to $n/(2n+4)$. Thus, the expected payment when the designer maximizes the highest effort must be smaller than the expected payment when the designer maximizes the total effort, otherwise, there is a contradiction to the optimality of the reward function in this example.

Also notice that the designer's profit in this case is

$$\int_0^1 [x(\theta) - R(x(\theta))] dF^n = \int_0^1 [\frac{\theta^{n+1}}{2} - \frac{2n+2}{4(2n+1)} \left(\theta^{n+1}\right)^{\frac{n+2}{n+1}}] d\theta^n = \frac{n}{4(2n+1)}.$$

This shows that the designer's profit is increasing in n and approaches 1/8.[15]

Why is the winning reward now increasing in effort? When the designer valued total effort, a worker with a lower type was valued independent of the number of contestants; however, here a lower-type worker has a lower chance of having the highest type when the number of contestants increase. Hence, his effort decreases in the number of contestants. As we see in the example, this effort does depend upon n, $(x(\theta) = \theta^{n+1}/2)$. This causes the probability of winning for a given effort x to approach $2x$. This means that we no longer have to compensate for a lower probability of winning by increasing rewards for lower efforts.

□ **Additively separable case.** Now we consider the case that the value function is additively separable.

Proposition 8. In the **highest-add** environment, the optimal reward is given by

$$R(x) = \left(c(\theta(x), x) + \int_{\theta^*}^{\theta(x)} [F(\theta)^{n-1} - c_\theta(\theta(\tilde{x}), \tilde{x}(\theta)] d\theta \right) / F(\theta(x))^{n-1} - \theta(x) \tag{15}$$

[15] The designer's profits does not always increase in n. For instance, let us say that all types have $\theta \approx 1$ and reward and costs are the same as in the example. Since there is little informational rent the designer keeps all the surplus. Obviously then, the designer does best by limiting participation to one and inducing $x(\theta) = 1/2$.

where $\theta(x)$ is the inverse function of the equilibrium effort $x(\theta)$ that is determined by

$$F(\theta)^{n-1} + c_{\theta x}(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)} = c_x(\theta, x(\theta)) \tag{16}$$

and the cutoff θ^* is the θ that maximizes the designer's profits from the set $\{\theta \in [\underline{\theta}, \overline{\theta}] : x(\theta)F(\theta)^{n-1} - c(\theta, x(\theta)) + \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)} = 0\}$ (if the set is empty $\theta^* = \underline{\theta}$).

The properties of the equilibrium efforts and the optimal reward in this case are derived as in the previous cases.[16]

Proposition 9. In the **highest-add** environment,

1. The equilibrium effort depends on the number of contestants.
2. Some contestants may not choose to participate in the contest.
3. The optimal reward is not always positive.
4. The optimal reward may be increasing for any number of contestants.

Example 4. In the **highest-add** environment, the cost function is $c(\theta, x) = x^2/\theta$ and the distribution of the contestants' types F is uniform on $[0, 1]$.

From this specification, we can rewrite (16) as $\theta^{n-1} - 2x(\theta)(1-\theta)/\theta^2 = 2x(\theta)/\theta$. This implies the equilibrium effort function

$$x(\theta) = \frac{\theta^{n+1}}{2}.$$

The inverse function is $\theta(x) = (2x)^{1/(n+1)}$. The cutoff equation is given by

$$\frac{\theta^{2n}}{4} + 2\theta^n - \theta^{n-1} = 0.$$

For $n = 2$, the solution of this equation that maximizes profits is $\theta^* = 0.486$. As $n \to \infty$, we have the profit maximizing solution as $\theta^* \to 0.5$. The optimal reward is then

$$R(x) = \frac{2n+2}{4(2n+1)}(2x)^{\frac{n+2}{n+1}} - (2x)^{\frac{1}{n+1}}\left(\frac{n-1}{n}\right) - (2x)^{-\frac{n-1}{n+1}}\left[\frac{\theta^{*n}}{n} + \frac{\theta^{*2n+1}}{4(2n+1)}\right].$$

For large n, this reward approaches to the increasing reward function $R(x) = x/2 - 1$.

As in mechanism design, the contestants earn their informational rents and the designer gets total surplus minus the informational rents. One may ask when the designer's objective is to

[16] Similar to before, we note that in the additively separable case, the equilibrium effort function when the designer values the expected highest effort is point-wise smaller than the equilibrium effort when the designer values the expected total effort. In addition, the set of types of contestants when the designer values the total effort is larger (and includes) than the set of types of contestants when the designer values the expected highest effort.

maximize the highest expected effort, why the designer cannot use a mechanism that elicits types beforehand and requires only the highest type to expend effort (similar to a first-price auction which we would not call a contest)? Doing so would save having to "reimburse" losers for their expenditure and thus be able to increase total surplus since the designer only benefits from the highest effort. Indeed, in some circumstances, the designer may have this ability, however, there are many reasons why he may not. For instance, the competition among the players sometimes is crucial to obtaining the full value to winning. This is clearly the case in any athletic contests. Furthermore, using a first-price auction may create credibility issues (such as renegotiation); once others are dismissed, the contestant will gain bargaining power against the designer. Finally, there could be a learning by doing component to private information. This would be a realistic component to technological contests.

In this paper, we find the optimal contest design (we distinguish contests from other mechanisms by their open all-pay nature).[17] We can claim it is an optimal design for a contest since we solve for the optimal induced effort and find a reward structure that induces it. While the reward structure studied thus far, an effort-dependent reward for first-place, is a suitable mechanism, it is just one of many optimal reward structures. In the next section, we find a particularly useful characterization of optimal reward structures.

5. Multiple Rewards

So far, we analyzed the optimal reward when an effort-dependent reward $R(x)$ is awarded only to the winner. We now extend the analysis to the case of multiple effort-dependent rewards where

[17] In other words, there can't be any preselection of contestants, all must be treated equally, and messages sent to the designer can only be done through expending effort. Formally, the mechanism is a contest if the expected payment to contestant i is a function only of the efforts, $p_i(x_1, ..., x_n)$ and is symmetric in i, $p_i(..., x_i, ..., x_j, ...) = p_j(..., x_j, ..., x_i, ...)$. This symmetry also prevents a designer from limiting the number of contestants. In many situations such as technological contests (or open math problems), it is realistic that the designer would lack this ability.

the contestant with the highest effort wins the reward $R_1(x)$, the contestant with the second highest effort wins the reward $R_2(x)$, etc., that is, $R_k(x)$ is the reward for the contestant with the k^{th}-highest effort who exerts an effort of x. The corresponding value functions for having the k^{th}-highest effort is $V_k(\theta, R_k(x))$. In this extended environment, we can ask what are the optimal structures and the optimal number of effort-dependent rewards. Moldovanu and Sela (2001), using our environmental assumption of convexity of the cost function, show that it may be optimal to allocate several rewards. Since their rewards were not only fixed but independent of effort, it is interesting to examine if their result holds in our environments where the designer has more flexibility. This brings us to the following proposition, in which the second environment corresponds to that studied in Moldovanu and Sela (2001).

Proposition 10. A multi-reward contest $\{R_i(x)\}_{i\geq 1}$ with equilibrium inverse effort function $\theta(x)$ has ex-ante equivalent payoffs (for both the designer and the contestants) to a single reward contest $R(x)$ with the same equilibrium inverse effort function if $E[R_i(x)|\theta(x)] = F(\theta(x))^{n-1}R(x)$ for the following environments (independent of the objective function of the designer):[18]

(i) the multiplicative-separable case, when a contestant of type θ with the $k^{th}-$highest effort has value of $V_k(\theta, R_k(x)) = R_k(x) \cdot \theta$.

(ii) the independent case (the payoff to winning is independent of θ), when a contestant of type θ with the $k^{th}-$highest effort has value $V_k(\theta, R_k(x)) = R_k(x)$.

(iii) the additively separable case, the contestant of type θ with the k^{th}-highest effort has value

$$V_k(\theta, R_k(x)) = \begin{cases} R_1(x) + \theta & \text{if } k = 1, \\ R_k(x) & \text{otherwise.} \end{cases}$$

Proof. See the Appendix.

Proposition 10 implies that any monotonic equilibria in an environment with either a single reward or multiple rewards can be induced in an environment with the other reward structure simply by defining the rewards such that $E[R_i(x)|\theta(x)] = F(\theta(x))^{n-1}R(x)$. In addition to having the same equilibrium effort functions, the ex-ante payoffs are the same with either reward structure.

[18]The expression $E\left[R_i(x)|\theta(x)\right]$ is the expected reward of each contestant given the equilibrium bid function. For example, if there is only one reward, this expected value is $F(\theta(x))^{n-1}R_1(x)$, and if there are two rewards, this expected value is $F(\theta(x))^{n-1}R_1(x) + (n-1)F(\theta(x))^{n-2}(1 - F(\theta(x)))R_2(x)$.

For this to happen, the three cases ensure that surplus is not created nor destroyed simply by the mere fact of giving a non first-place reward – there is no intrinsic value to being runner-up.[19]

The proof of the proposition comes from the fact that both $E[R_i(x)|\theta(x)]$ and $F(\theta(x))^{n-1}R(x)$ are substitutable in both the contestant's expected surplus and the designer's expected profits. Their equivalence implies the equivalence of the contestants' payoffs between the two possible reward structure. Since the rewards are eliminated by substituting the contestant's surplus into the designer's surplus, they do not appear in or affect the form of the designer's expected payoff that we maximize (to solve for the optimal effort). This leads us to the following corollary of the proposition.

Corollary 1. In all three cases of Proposition 10, there exists an optimal reward structure for rewards on any particular set of places. Furthermore, for each possible optimal reward, in addition to the designer's expected payoff, the contestants' payoffs and equilibrium effort function are also the same.

An interesting result of Corollary 1 is that one possible reward structure is that payment from the designer can be made independent of place. In other words, the first-place winner exerting effort x receives the same payment as he would if he came in second place. (In case (iii), there is still an extra value to coming in first.) In the independent case where the designer cares about total effort, this reduces to the principal-agent solution.

Proposition 10 allows us to easily analyze a range of problems including those with a disadvantage to the winner: everyone may want to try to beat the fastest gunfighter in town, the tallest building may be a clearer target for terrorism, being the expert academic in a field may invite more referee reports, etc.[20] In the following example we generate a peculiar example of an optimal

[19]Some other possibilities do not share this property: For example, in the additively separable case if a contestant of type θ with the i-highest effort receives $R_i(x) + \theta$ for all i. In this case, the equivalence would disappear since giving additional rewards (say of value ϵ) would create surplus.

[20]Donald Trump is quoted as saying "Nobody is going to want to live in a building that's a target," in reference to why the proposed 115-story condominium Fordham Spire in Chicago is not economically viable. Note that he is constructing a mere 92-floor condominium skyscraper there. (July 26, 2005, Associated Press).

contest with two contestants where there is only a reward for the loser.[21]

Example 5. Consider the independent case with two contestants where the designer maximizes the total effort. The cost function is $c(\theta, x) = x^2/\theta$ and F is uniform on $[0, 1]$.

In this example we have shown that the optimal reward function for the winner is $R(x) = 2x/3$ and the equilibrium effort is given by $\theta(x) = \sqrt{2x}$. We then have $F(\theta(x))^{n-1}R(x) = (2x)^{3/2}/3$. One can maintain the same revenue by giving a reward of zero to the "winner" and an effort-dependent reward to the loser. This would be set such that $(1 - \theta(x))R_2(x) = (2x)^{3/2}/3$. Thus, the optimal rewards are

$$R_1(x) = 0, \quad R_2(x) = \frac{(2x)^{3/2}}{3\left(1 - \sqrt{2x}\right)}.$$

Notice that this reaches infinity as $x \to 1/2$ (this is the effort chosen by the highest type, $\theta(1/2) = 1$), since there is an almost certain chance of winning and getting paid nothing.

While we examined rewards that depended only on one's own effort, we can also allow the rewards to depend upon the vector of efforts (this can be thought of as introducing externalities). For instance, if there are two contestants, the reward to contestant i for winning could be $R(x_i, x_j) = x_i - x_j$. In much the same manner as the case of multiple rewards, the reward function is simply a means for inducing the optimal effort. Thus, given induced equilibrium effort $x(\theta)$ and optimal reward function $R(x)$, any multi-effort reward function $R(x_1, ..., x_n)$ is also optimal if $E[R(x_1(\theta_1), ..., x_i, ...x_n(\theta_n))] = R(x_i)$ for all x_i.

We demonstrate the above by using the environment of example 5. Before we found that the optimal reward is $R(x) = 2x/3$ and the equilibrium effort is $\theta(x) = \sqrt{2x}$. Using this solution as a basis, an optimal reward to contestant i for winning can be either $R(x_i, x_j) = x_i/3 + \sqrt{2x_j}/3$ or $R(x_i, x_j) = x_i - \sqrt{2x_j}/3$ (since $E[\sqrt{2x_j}/3|x_j < x_i] = \int_0^{\sqrt{2x_i}} \theta/3 d\theta = x_i/3$). From this it is interesting to see that rewards can be both increasing and decreasing in the opponent's effort. We also see that the reward can be negative (if $x_j = x_i - \varepsilon = 1/8$, then the latter reward function equals $1/8 + \varepsilon - 1/6$). This is perhaps surprising since it is the independent case and the value to winning is equal to the rewards of winning. Hence, even ignoring cost of effort, each contestant still

[21]We can also easily generate other examples such as a three-player contest where the player with the mid-level effort gets the reward. Or even, with three players there is a reward for the winner and a consolation prize for the loser (the mid-level effort player is the only one not getting a reward).

(weakly) makes a loss (for a particular draw of types). Moreover, even ignoring the benefit from efforts, the designer makes a profit. Finally, we remark that in the same manner, multiple rewards can be combined with these multiple effort rewards.

6. Concluding remarks

■ In this paper, we study the design of contests when the designer has full flexibility over what reward function to use. We solve our problem of finding the optimal reward by indirect means. We start by solving for the optimal effort function. This is done by looking at the virtual cost of increasing effort for a specific contestant type. Then, we solve for the reward that induces this effort function. Using this method, we analyzed two objective functions for the designer and two value functions of the contestants. Our results from this analysis are summarized in Table 1.

From Table 1 and the results in section 5, our main findings are as follows. First, the optimal reward may either increase or decrease in the contestants' effort. Second, the optimal reward may also be negative. Third, the optimal reward does not necessarily eliminate participation of the contestants with the lowest types. Fourth, it does not matter upon how many rewards the optimal reward is distributed.

The most surprising result is that the reward to winning may be not only increasing, but decreasing in the efforts. It is easy to envision contests where the reward to winning is increasing in the results. These bonuses for good performances may be external rewards to winning, extra payment from the designer, or simply getting the reward sooner. On the other hand, a scenario where the reward is actually decreasing in effort may not be obvious. However, it does occur in the case of contests where the reward is increasing over time. This can happen if the money for winning a contest similar to the X-prize is increased over time by having the organizers continue to raise funds from sponsors. The reason that this is in fact a decreasing reward is that inventing early requires a greater effort. Thus, time and effort are in opposite directions and while the reward is increasing over time, it is decreasing in effort.

While the environment we study here is restricted to contests, it is possible to use the same tools to study optimal design with effort-dependent rewards in other environments. For example, one can study the classic auction mechanisms with effort-dependent rewards. Alternatively, one can study a hybrid model where only part of the effort is sunk and the rest is expended after a winner is selected such as in contests for architectural contracts.

7. Appendix

■ Proofs to Propositions 1–6, 10 and Corollary 1 follow.

Proof of Proposition 1. Using the envelope theorem on the contestant's maximization problem (1) yields

$$\pi'(\theta) = F(\theta)^{n-1}V_\theta(\theta, R(x(\theta))) - c_\theta(\theta, x(\theta)).$$

Assume that all contestants with type $\theta \geq \underline{\theta}$ take part in the auction and that $\pi(\underline{\theta}) = 0$. Then by integration we obtain

$$\pi(\theta) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1}V_\theta(\tilde{\theta}, R(x(\tilde{\theta}))) - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta}.$$

From the maximization problem, we also have

$$\pi(\theta) = F(\theta)^{n-1} \cdot V(\theta, R(x(\theta))) - c(\theta, x(\theta)).$$

Combining these two equations yields the desired result. *Q.E.D.*

Proof of Proposition 2. Straightforward substitution of (5) into (2) implies that an equilibrium strategy $x(\theta)$ must satisfy

$$F(\theta)^{n-1}R(x(\theta)) - \hat{c}(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} -\hat{c}_\theta(\tilde{\theta}, x(\tilde{\theta}))d\tilde{\theta}. \quad \text{(A1)}$$

Substituting (A1) in the designer's expected payoff (3) yields the following designer's problem

$$\max_x n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - \hat{c}(\theta, x(\theta)) + \int_{\underline{\theta}}^{\theta} \hat{c}_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta} \right] dF. \quad \text{(A2)}$$

By using integration by parts, we can rewrite the last term as follows

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \hat{c}_\theta(\widetilde{\theta}, x(\widetilde{\theta}))d\widetilde{\theta}dF = \int_{\underline{\theta}}^{\overline{\theta}} \hat{c}_\theta(\theta, x(\theta))d\theta - \int_{\underline{\theta}}^{\overline{\theta}} F(\theta)\hat{c}_\theta(\theta, x(\theta))d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \hat{c}_\theta(\theta, x(\theta))\frac{1 - F(\theta)}{f(\theta)}dF.$$

Thus, the designer's problem is

$$\max_x n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - \hat{c}(\theta, x(\theta)) + \hat{c}_\theta(\theta, x(\theta))\frac{1 - F(\theta)}{f(\theta)} \right] dF.$$

Since the designer is indirectly choosing $x(\theta)$ through the reward function, we can look at the first-order condition to find the induced optimal effort

$$1 + \hat{c}_{\theta x}(\theta, x(\theta))\frac{1 - F(\theta)}{f(\theta)} = \hat{c}_x(\theta, x(\theta)). \quad \text{(A3)}$$

It is straightforward to show that our assumptions on c imply the same assumptions on $\hat{c} = \frac{c}{\theta}$ for all $\theta > 0$. (The reverse isn't true.) Now note that $1 + \hat{c}_{\theta x}(\theta, 0)\frac{1-F(\theta)}{f(\theta)} > \hat{c}_x(\theta, 0)$ for all $\theta > 0$. Our assumptions also imply that when x increases, the LHS of (A3) decreases and the RHS strictly increases. The limit has the property that $\lim_{x\to\infty} 1 + \hat{c}_{\theta x}(\theta, x)\frac{1-F(\theta)}{f(\theta)} < \lim_{x\to\infty} \hat{c}_x(\theta, x)$. Thus, a unique interior solution exists. In addition as θ increases (keeping x fixed) the LHS of (A3) increases and the RHS decreases. Hence, there is a unique strictly increasing solution to this equation. Furthermore, since $\hat{c}$ is thrice continuously differentiable, $\hat{c}_{\theta^2 x}$ $\hat{c}_{x\theta}, \hat{c}_{x^2}$, and $\hat{c}_{\theta x^2}$ are finite for all interior θ. This then implies that $x(\theta)$ is continuous for all $\theta > 0$. We should also note that second-order conditions are satisfied, since the designer's problem is strictly concave, because $-\hat{c}_{xx}(\theta, x) + \hat{c}_{\theta x^2}(\theta, x)\frac{1-F(\theta)}{f(\theta)} < 0$ for all $x \geq 0$.

Given the optimal effort $x(\theta)$, the optimal reward is obtained by changing variables from θ to x in equation (A1). Therefore, the optimal reward is simply

$$R(x) = \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x})d\theta(\tilde{x})\right)/F(\theta(x))^{n-1}$$

where $\theta(x)$ is the inverse of $x(\theta)$. *Q.E.D.*

Proof of Proposition 3. Here, we prove point 4 of Proposition 3. The optimal reward given by (6) can be written as a fraction of two strictly positive functions, $z_1(x)/z_2\,(x)^{n-1}$ where

$$\begin{aligned} z_1(x) &= \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x})d\theta(\tilde{x})\right), \\ z_2(x) &= F(\theta(x)). \end{aligned}$$

The derivative of the reward function is given by

$$\frac{z_2\,(x)^{n-2}\left[z_2\,(x)\,z_1'(x) - (n-1)z_2'\,(x)\,z_1(x)\right]}{z_2\,(x)^{2n-2}}.$$

Since by our assumptions $z_1\,(x)\,,\ z_2\,(x)\,,\ z_2'\,(x)$ are positive and finite, for large enough n, this derivative must be negative. *Q.E.D.*

Proof of Proposition 4. The equilibrium strategy $x(\theta)$ is given by the implicit function

$$F(\theta)^{n-1}[\theta + R(x(\theta))] - c(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta} \qquad \text{(A4)}$$

while the expected payoff of a contestant given this strategy is

$$\pi(\theta) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta}.$$

As before, we can use (A4) to find the reward as a function of the equilibrium effort

$$R(x) = \left(c(\theta(x), x) + \int_0^x [F(\theta(\tilde{x}))^{n-1} - c_\theta(\theta(\tilde{x}), \tilde{x})]d\theta(\tilde{x})\right)/F(\theta(x))^{n-1} - \theta(x).$$

By substituting this reward into the designer's payoff and using integration by parts, we obtain

$$
\begin{aligned}
& n\int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - c(\theta, x(\theta)) + \theta F(\theta)^{n-1}\right] dF - n\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \left[F(\tilde{\theta})^{n-1} - c_\theta(\theta, x(\theta))\right] d\tilde{\theta} dF \\
= \; & n\int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - c(\theta, x(\theta)) + \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)}\right] dF. \qquad \text{(A5)}
\end{aligned}
$$

The first-order condition of this yields the optimal effort function

$$1 + c_{\theta x}(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)} = c_x(\theta, x(\theta)).$$

As before, our assumptions on c guarantee that the designer's problem is strictly concave, satisfying the second-order conditions. The designer also has the option of having a cutoff type in order to not include lower types for when the expression within the integral is negative. It is important to notice that this expression within the integral does not necessarily increase in θ. *Q.E.D.*

Proof of Proposition 5

Here we prove point 4 of Proposition 5. Since the optimal equilibrium effort is the same as in the multiplicatively separable case when the value function is type-independent, the difference between the two rewards is that now the reward is larger by

$$\int_0^x [F(\theta(\tilde{x}))^{n-1} d\theta(\tilde{x}) / F(\theta(x))^{n-1} - \theta(x).$$

The derivative of this with respect to x is

$$
\begin{aligned}
& \frac{F(\theta(x))^{2n-2}\theta'(x) - \int_0^x [F(\theta(\tilde{x}))^{n-1} d\theta(\tilde{x}) \cdot (n-1)F(\theta(x))^{n-2}F'(\theta(x))\theta'(x)}{F(\theta(x))^{2n-2}} - \theta'(x) \\
= \; & \frac{-\int_0^x [F(\theta(\tilde{x}))^{n-1} d\theta(\tilde{x}) \cdot (n-1)F'(\theta(x))\theta'(x)}{F(\theta(x))^{n}} < 0.
\end{aligned}
$$

Thus, the reward is also decreasing for large n. *Q.E.D.*

Proof of Proposition 6. As in the case of maximization of total effort, we can use equation (A1) to substitute for $F(\theta)^{n-1}R(x(\theta))$ in the designer's expected payoff (12) and use integration by parts to simplify. Now the designer's expected payoff becomes

$$n\int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta)F(\theta)^{n-1} - \hat{c}(\theta, x(\theta)) + \hat{c}_\theta(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)}\right] dF. \qquad \text{(A6)}$$

The first-order condition of this yields the optimal (profit-maximizing) $x(\theta)$

$$F(\theta)^{n-1} + \hat{c}_{\theta x}(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)} = \hat{c}_x(\theta, x(\theta)). \qquad \text{(A7)}$$

Since $F^{n-1}(\theta)$ is increasing in θ, the same arguments as before guarantees a monotonic solution. From equation (A1), we find the optimal reward:

$$R(x) = \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x}) d\theta(\tilde{x})\right) / F(\theta(x))^{n-1} \qquad \text{(A8)}$$

where $\theta(x)$ is the inverse of $x(\theta)$ that satisfies (A7). *Q.E.D.*

Proof of Proposition 10. With multiple rewards, instead of equations (A1) and (A4), a contestant's expected surplus equation needs to be written as one of the following two equations (the first holds for cases (i) and (ii), while the second holds for case (iii)):

$$E[R_i(x(\theta))] - \hat{c}(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} -\hat{c}_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta},$$

$$F(\theta)^{n-1}\theta + E[R_i(x(\theta))] - c(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta}.$$

Since $E[R_i(x(\theta))] = F(\theta)^{n-1}R(x(\theta))$, a contestant's surplus does not change for a given effort. The designer's payoff changes from equations (3) and (12) to the following two formulas, respectively,

$$n\int_{\underline{\theta}}^{\overline{\theta}} x(\theta)dF - n\int_{\underline{\theta}}^{\overline{\theta}} E[R_i(x(\theta))]dF,$$

$$\int_{\underline{\theta}}^{\overline{\theta}} x(\theta)dF^n - n\int_{\underline{\theta}}^{\overline{\theta}} E[R_i(x(\theta))]dF.$$

When we use the contestants' surplus equations to substitute for the expected rewards in the above two formulas (depending upon the case and whether the designer's goal is total or highest effort), we arrive at exactly the same formulas for the designer's payoff as before: (A2), (A5) and (A6). Thus, both the induced effort and the respective payoffs will remain the same. *Q.E.D.*

Proof of Corollary 1. For any $\{R_i(x(\theta))\}$, one can find a $R(x(\theta))$ such that $E[R_i(x(\theta))] = F(\theta)^{n-1}R(x(\theta))$. Furthermore, for any $R(x)$, and nonempty set of places (for example, first place, second place and fourth place) one can find a $\{\widetilde{R}_i(x(\theta))\}$ such that $E[\widetilde{R}_i(x(\theta))] = F(\theta)^{n-1}R(x(\theta))$ and $\widetilde{R}_i(x) = 0$ for all i not in that set. This and Proposition 10 imply that if we find a optimal reward structure for a particular set of places, we can find the optimal reward structure on another set of places that yields the same expected payoffs to the designer and contestants (and same equilibrium effort function). Hence, any of these would be an optimal reward structure overall. *Q.E.D.*

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Table 1: Summary of Results

	$V = \theta + R$	$V = \theta \cdot R$
Total Effort	Induced effort $x(\theta)$ is independent of n $R'(x) < 0$ for large n Reward may be negative Some stay out of contest	$x(\theta)$ is independent of n $R'(x) < 0$ for large n Reward is positive All contestants participate
Maximum Effort	Induced effort $x(\theta)$ depends on n May have $R'(x) > 0$ for large n Reward may be negative Some stay out of contest	$x(\theta)$ depends on n May have $R'(x) > 0$ for large n Reward is positive All contestants participate