Power and Core-Periphery Networks*

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Abstract

The heterogeneous connections model is a generalization of the homogeneous connections model of Jackson and Wolinsky (1996) in which the intrinsic value of each connection is set by a discrete, positive and symmetric function that depends solely on the types of the two end agents. Core periphery networks are defined as networks in which the agents' set can be partitioned into two subsets, one in which the members are completely connected among themselves and the other where there are no internal links. A two-type society is defined as "power based" if both types of agents prefer to connect to one of the types over the other, controlling for path length. An exhaustive analysis shows that core periphery networks, in which the "preferred" types are in the core and the "rejected" types are in the periphery, are crucial in the "power based" society. In particular, if the linking costs are not too low and not too high, at least one such network is pairwise stable. Moreover, in many cases these networks are the unique pairwise stable networks and in all cases they are the unique strongly efficient networks. The set of efficient networks often differs from the set of pairwise stable networks, hence a discussion on this issue is developed. These results suggest heterogeneity accompanied by "power based" linking preferences as a natural explanation for many core-periphery structures observed in real life social networks.

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1 Introduction

A network is defined to be a core-periphery network if its set of agents can be partitioned into two subsets, the core and the periphery, such that each agent in the core is directly connected to all other core members while each periphery member is directly connected to none of the other periphery agents[1]. In this paper we introduce a simple network formation model in which core periphery networks are the dominant architecture both as stabile networks and as efficient networks.

Since the 1970's the empirical literature of social networks identified core periphery architecture as a dominant social structure in many contexts[2]. Core-periphery structures were found in macroeconomics in the theory of spatial division of production (Krugman (1991, 1994) and Fujita et. al (2001)) and in the sociology-oriented world system literature[3] (Wallerstein (1974), Chase-Dunn and Grimes (1995) and Smith and White (1992)). These architectures were found also in industrial organization, both in general, in the analysis of firms' power structure as reflected in the interlocking directorates' network (Mintz and Schwartz (1981a, 1981b)) and in specific industries as the airline industry (Starr and Stinchcombe (1992)) and the local and long distance phone calls industry (Economides (1996)). Core-periphery structures were found in formal and informal social organizations as factions and other quasi-groups based on recruitment by existing members (Boissevain (1968)), solidarity networks with asymmetries in wealth and status (Fafchamps (1992)), scientific networks (Brieger (1976), White et. al. (1976), Mullins et. al. (1977),

[1] The core-periphery structure is not a well-defined concept in the social sciences literature. Indeed, most of the researches that use this phrase mean that there is one group of agents that is densely connected internally, while all the other agents are sparsely connected internally (Borgatti and Everett (1999)). The definition here is identical to the one in Bramoulle and Kranton (2003) and Bramoulle (2007). However, network is a core-periphery network by Goyal (2007) if the set of agents can be partitioned into two subsets, core and periphery, such that each agent in the core is directly connected to all other core members and each periphery member has a single link to one of the core members. Galeotti and Goyal (2008) restrict the pattern of links in a core periphery network to be complete – every peripheral agent is connected to all core agents. Later we will refer to the definition by Goyal (2007) as minimally connected core periphery networks and to the one by Galeotti and Goyal (2008) as maximally connected core periphery network. In the mathematical graph theory literature core periphery networks are called split graphs (Foldes and Hammer (1977) and Brandstadt et. al (2004)).

[2] White et. al. (1976) mention that one of the frequent structures they encounter has one group internally connected and one group internally disconnected which are reciprocally connected between them.

[3]The theory states that national development could only be understood as the complex outcome of local interactions with an expanding world economy. Further, the world countries have hierarchical power order of core, semi periphery, and periphery that is reflected both in world economy and in international relations. The core countries are stronger (e.g. military power) then others and exploit the weak periphery countries either by tributes or by favorable market conditions. Therefore the core countries can be distinguished by their internal massive volume of trade and by their capital-intensive production.

Granovetter (1983), Grossman and Ion (1995), van der Leij and Goyal (2006)), internal firms' networks (Krackhardt and Hanson (1993)[4]) and in the social network of injecting drug users (Curtis et. al. (1995)).

In most of the empirical examples mentioned above, it is evident that the members of the core have some intrinsic advantage over the members of the periphery – either the financial institutions that are positioned in the core of the directorates' network, the veteran members in factions or the eminent scientists in the scientific networks. In many cases these advantages do not initially stem from these core members' position in the network, but they lead their possessors to be extremely central in the social network. We suggests that in order for the advantageous agents to be placed in a central position they have to be recognized as more attractive by all the members of the community, advantageous and disadvantageous. This recognition is the main source of power of the advantageous agents. Once these agents are placed in a central position in the social network, their advantage can be reinforced and perpetuated[5].

Some network formation models in the social sciences literature[6] might suggest explanations for the formation of core-periphery structures. Models associated with the structural balance theory, are meant mainly to explain various segregation architectures. Therefore, these models have to assume some internal animosity among the periphery members in order to explain the sparse internal network attributed to the periphery[7]. Models associated with preferential attachment (also known as degree variance model) need to assume that core members preceded the periphery members in the network. The extended preferential attachment model of Bianconi and Barabasi (2001) adds heterogeneity in the form of fitness into the links accumulation process

[4] Krackhardt and Hanson (1993) consider the core-periphery structure as problematic for the firm since it signals that many workers depend on a small group of central agents. Borgatti (2005) considers it as favorable structure as efficient spreader of knowledge. However, he points out that since the core controls the content of the knowledge, these networks might not be good at innovation because it makes it is easy for the conventional wisdom to swamp new ideas (see also Chubin (1976), Granovetter (1983) and Bramoulle and Kranton (2003)).

[5]Brieger (1976) and White et. al. (1976) found a hierarchy of statuses in the scientific network, where the upper "class" was known by all the lower strata but unaware of most of them. The internal awareness of the lower "classes" was partial. Brieger (1976) clarifies (in a footnote) that the term "status" refers to differentiation of persons on some vertical continuum of "prestige" or "power".

[6] See Banks and Carley (1996) for a short survey of the main network formation theories in sociology and Goyal (2007) and Jackson (2008) for network formation models in economics. See Newman (2003) for a survey of networked systems models in physics.

[7] In this theory, the social structure is a graph in which each link is one or more relations between two nodes where a signed number describes each relation. The value of a link is the sum of these numbers and the value of a cycle is the multiplication of its links values. The benefit of a person from a graph is the sum of values of all the circles that go through him. In the basic version, a cycle is balanced if and only if its value is positive and a social network is balanced if and only if all its cycles are balanced. Since people maximize their values, the theory argues that social networks that are balanced will be stable. See Heider (1946, 1958), Cartwright and Harary (1956), Newcomb (1956, 1961), Davis (1963, 1967), and Doreian and Mrvar (1996).

and thus enables very fit agents to have higher degree than older, but not as fit, agents. Moreover, in order to generate the cohesiveness of the core, probably some rewiring should be introduced on top of the heterogeneity.

Our framework is a generalization of the homogenous connections model of Jackson and Wolinsky (1996) to accommodate two types. As in the original model, the agents benefit from their direct connections (costly) and indirect connections. However, the benefits are a function of the two end agents (the intrinsic value of the connection) and the distance between them. In the homogeneous model, the star, a degenerative form of a core periphery network, appears as a dominant architecture. However, the star network is stable and efficient, independently of the central agent and therefore the results cannot be interpreted as a process of power perpetuation by central positioning in the social network.

Some models introduced heterogeneity to the connections model of Jackson and Wolinsky (1996) through the linking costs rather than through the intrinsic values[8]. The important difference between these two approaches is that the linking costs heterogeneity is relevant only to direct connections, while the intrinsic values are carried through both direct and indirect connections[9]. Indeed, it turns out that none of the versions of the connections model which introduced heterogeneous linking costs exhibit core periphery networks as either stable or efficient. Moreover, core periphery networks in which there are more than two agents in the core were not found to be Nash networks in the various versions of the one-sided model of Bala and Goyal (2000)[10].

[8] See Johnson and Gilles (2000), Jackson and Rogers (2005) and Carayol and Roux (2005). Note that core periphery networks might arise for certain parameters in the two-islands model of Jackson and Rogers (2005) if the internal linking costs of one island were lower from the external linking costs while the internal linking costs of the other island were higher from the external linking costs.

[9] This issue was approached also by Galeotti (2006) and Galeotti et. al. (2006), which introduced heterogeneity in both costs and benefits to the one-sided one-flow and the one-sided two-flow formation models of Bala and Goyal (2000), respectively. They find that cost heterogeneity affects both the connectedness and the architecture of the Nash networks. However, in the one-flow model the value heterogeneity affects both the connectedness and the architecture, while in two-flow it affects only the connectedness. We, on the other hand, find no effect of heterogeneity on the connectedness and a significant effect on the architecture.

[10] See Galeotti (2006), Galeotti et. al. (2006), Hojman and Szeidl (2008) and Feri (2007). Core periphery networks cannot be stable also in the framework of McBride (2006) unless possibly under certain parameters in the case where the agents know only their direct friends in the network. Hojman and Szeidl (2006) show the conditions under which a socially "gifted" agent becomes the center of their stable star architecture. However, it seems hard to extend this example to core periphery networks. Zeggelink (1995) introduces a network formation model with two types of agents. In this model the agents' loss depends on her deviation from her exogenous ideal state which is characterized by an ideal number of friends, all of them are similar to her. These myopic agents take part in a dynamic process, in which friendship connection must be reciprocated, until they reach as near as possible to their ideal position. However, none of the simulations of this model generated a core-periphery network.

Galeotti and Goyal (2008) suggest a homogeneous explanation for the formation of core periphery networks. In their model, an agent can either acquire information personally or gather information from agents that acquired it personally. They show that if information could be gathered only directly from one of the agents that acquired it personally, every stable network is a core periphery network where the core includes the agents that acquired the information personally and the periphery include the agents that need to gather the information through the network.

In what follows we will introduce heterogeneity into the connections model of Jackson and Wolinsky (1996) in order to analyze the case in which all the members of the society acknowledge the advantage of one of the types and therefore prefer linking to agents of this type over other agents. Under this setting of unanimous preferences towards the advantageous type, we will show that core periphery networks are both pairwise stable (unique in many cases) and uniquely efficient and discuss cases of tension between these two concepts.

The next section will introduce the heterogeneous connections model and define a "power based" society. It will also define several special architectures of core periphery networks that will become useful in the analysis. The third section will give a complete characterization of the stable and efficient networks of the "power based" society to show that core-periphery structures play a major role in this context. The last section will conclude with a detailed interpretation of the results and some natural and possible future research directions.

2 The Model

Preliminaries

Consider a finite set $N = \{1,2,\ldots,n\}$ of utility-maximizing agents. The *complete network*, g^N, is the set of all subsets of *N* of size two, while the *empty network* is the empty set. The set of all possible networks on *N* is $\{g \mid g \subseteq g^N\}$. Denote by *ij* the element of g^N that contains *i* and *j*. If $ij \in g$ we say that agents *i* and *j* are *directly connected* in network *g*. Denote by $N(i,g) = \{j \in N \mid ij \in g\}$ the set of agent *i*'s *neighbors* in network *g*. Let $g + ij$ denote the network obtained by adding the link *ij* to the network *g* and let $g - ij$ denote the network obtained by severing the link *ij* from the network g. A *path p* of length $L(p)$ between agent *i* and agent *j* in network *g* is a set of distinct nodes $\{i_1, i_2, i_3, \ldots, i_{L(p)}, i_{L(p)+1}\}$ such that $\{i_1 i_2, i_2 i_3, \ldots, i_{L(p)} i_{L(p)+1}\} \subseteq g$ and $i_1 = i, i_{L(p)+1} = j$. Let $k \in p$ and denote the *position* of agent *k* in path *p* by $t^k(p)$, meaning, $t^k(p) = x \Leftrightarrow i_x = k$. If a path between agent *i* and agent *j* exists in network *g*, we say that agent *i* and agent *j* are *connected* in network *g*. Otherwise, we say that agent *i* and agent *j* are *disconnected* in network *g*. If agent *i* and agent *j* are connected but not directly connected in network g, we say that agent *i* and agent *j* are *indirectly connected* in network g. For a subset of the agents' set $N' \subseteq N$, define a *subnetwork* g' to be the set of all pairs of agents $i, j \in N'$ such that $ij \in g$. The subnetwork $g' \subseteq g$ is a *component* of network *g* if for all pairs of agents $i, j \in N'$, agent *i* and agent *j* are connected in *g'* and there is no pair of agents $i \in N', j \in N - N'$ such that $ij \in g$. Denote by $\tilde{N}(i,g)$ the set of agents that reside in the same component as agent *i* in network *g*. If for each pair of agents $i, j \in N$, agent *i* and agent *j* are connected in *g*, we say that *g* is *connected*. A path *p* between agent *i* and agent *j* in network *g* is a *shortest path* between those agents if there is no other path p' between them such that $L(p') < L(p)$. Denote the set of all shortest paths between agent *i* and agent *j* in network *g* by $S(i,j,g)$, its cardinality by s_{ij} and the path's length by d_{ij}. Let $S^k(i,j,x,g) = \{s \in S(i,j,g) \mid k \in s, t^k(s) = x\}$ be the set of all shortest paths between agent i and agent j in network g such that agent k is in position x and denote its cardinality by $s_{ij}^{k}(x)$[11].

The homogeneous symmetric connections model without side payments

Jackson & Wolinsky (1996) introduces the homogeneous symmetric connections model without side payments. In this model, the utility of agent i from network g is $u_i(g) = \sum_{j \neq i} \delta^{d_{ij}} - \sum_{j:ij \in g} c$ where $0 < \delta < 1$ captures the idea that the value that agent i derives from being connected to agent j is proportional to their proximity and $c > 0$ is the universal direct connection costs[12]. The network g is *pairwise stable* with respect to the utility function if for every existing link, both its agents would not gain by severing it ($\forall ij \in g : u_i(g) \geq u_i(g - ij), u_j(g) \geq u_j(g - ij)$) and for every non-existing link, either at least one of its agents strictly loses from forming it or both agents do not gain from forming it ($\forall ij \notin g : u_i(g + ij) > u_i(g) \Rightarrow u_j(g) > u_j(g + ij)$). The network g is *strongly efficient* if there is no other network on N for which the sum of utilities (denoted by $v(g)$) is higher ($\forall g' \neq g : v(g) \equiv \sum_{i \in N} u_i(g) \geq \sum_{i \in N} u_i(g') \equiv v(g')$). A *star network* is a network in which there is a central agent who is directly connected to all other agents in N while these other agents are connected directly only to her. propositions 1 and 2 in Jackson and Wolinsky (1996) characterizes stability and efficiency in the homogeneous symmetric connections model by identifying four possible relations between the linking costs and the depreciation factor[13]. When costs are very low ($c < \delta - \delta^2$) the unique pairwise stable network and the unique strongly efficient network is the complete network. When the costs are intermediate ($\delta - \delta^2 < c < \delta$) the star network is pairwise stable (but not unique) and the unique

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[11] Note that $\forall x \leq d_{ij} + 1, \forall j \in \tilde{N}(i, g): \sum_{k \in \tilde{N}(i,g)} s_{ij}^{k}(x) = s_{ij}$ and that $s_{ij}^{k}(x) > 0 \Rightarrow \forall x' \neq x : s_{ij}^{k}(x') = 0$.

[12] The optimization problem of the individual in this model can be interpreted as some kind of centrality maximization problem under costs constraint. It departs from the common centrality measures both by considering linking costs and by using an exogenous depreciation parameter (although similar concepts of distance depreciation appear in the closeness centrality measure, the information centrality measure and the attenuation parameter suggested first by Katz (1953) and used by Bonacich (1987) and many others). This model is very simple and therefore entails some strong assumptions as centrality maximization (see Shimbel (1953) for reservations), positive externalities (see the coauthors model in Jackson and Wolinsky (1996) for negative externalities) and shortest paths as the only source of utility (for reservations see Stephenson and Zelen (1989)).

[13] See Jackson (2008) for similar results given a more general distance-based benefit function.

strongly efficient network. When the costs are high ($\delta < c < \delta + \frac{n-2}{2}\delta^2$) the empty network is pairwise stable (each agent in any other pairwise stable network has at least two links), while the star network is the unique strongly efficient network (but, obviously, not pairwise stable). Last, when the costs are extremely high ($\delta + \frac{n-2}{2}\delta^2 < c$) the empty network is pairwise stable and the unique strongly efficient network. Later, we will use the fact that nothing in these results changes if the utility function of the agent is $u_i(g) = \sum_{j \neq i} A\delta^{d_{ij}} - \sum_{j:ij \in g} c$ for a positive constant A[14].

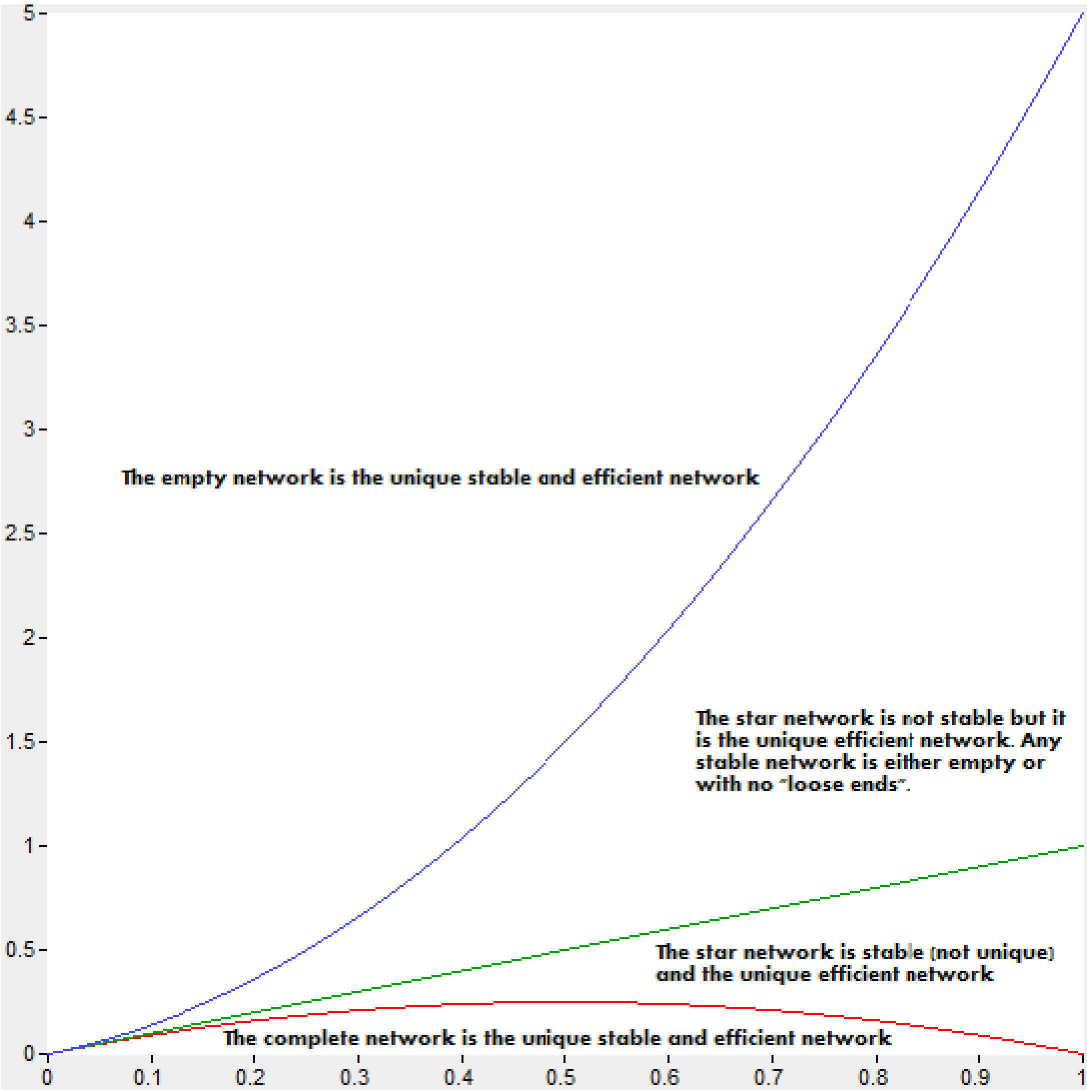

Figure 1: Graphical summary of propositions 1 and 2 in Jackson and Wolinsky (1996) for the case of *n=10*. The X-axis is the depreciation rate (δ) and the Y-axis is the linking costs (c).

[14] Mathematically, instead of accounting for the linking costs in the various cases of these propositions, one should refer to the linking costs normalized by the parameter, meaning to $\frac{c}{A}$.

The heterogeneous symmetric connections model without side payments

We allow for two types of agents in the framework described above, such that there are $k > 0$ type *a* agents and $l > 0$ type *b* agents ($k + l = n$). The agent's utility from each connection is a function both of her proximity to the other agent (as in the homogeneous model) and of the intrinsic value that this agent provides her[15]. Thus, the utility of agent *i* from network *g* is $u_i(g) = \sum_{j \neq i} \delta^{d_{ij}} f(t_i, t_j) - \sum_{j:ij \in g} c$ where $t_i \in \{a, b\}$ and $f(t_i, t_j)$ is the intrinsic value function. We assume that the intrinsic value function is symmetric, positive and depends only on the types of the agents: $f(t_i, t_j) = \begin{cases} w_1 & t_i = t_j = a \\ w_2 & t_i \neq t_j \\ w_3 & t_i = t_j = b \end{cases}$ (Jackson and Wolinsky (1996) use $w_3 = w_2 = w_1 = 1$).

The intrinsic value function might be interpreted as inducing a social norm regarding the benefit from connections in the society. In this paper we will concentrate on the case in which $w_1 > w_2 > w_3$. In this case, both types prefer a connection to an agent of type *a* over a connection of the same length to an agent of type *b*[16]. Therefore, we call a society with such values of the intrinsic value function a "power based" society since the agents' preferences could be interpreted as an attraction towards the powerful[17]. Note that type *a* is the preferred type for exogenous reasons, and in particular, for reasons which are independent from the network structure. For future use denote $w_{\min} = \min\{w_1, w_2, w_3\}$ and $w_{\max} = \max\{w_1, w_2, w_3\}$.

Core-periphery

A network *g* is a *core-periphery network* if there is a partition of the set of agents into two subsets *K* (the "core") and *L* (the "periphery") such that $K \cup L = N$, $K \cap L = \phi$ and $\forall i, j \in K : ij \in g$ while $\forall i, j \in L : ij \notin g$. Various classes of core-periphery

[15] This is the term used by Jackson and Wolinsky (1996) while describing the general connections model.

[16] Since the function is symmetric one can interpret these weights as strength of ties in the sense of Granovetter (1973). The highest value reflects both power and homophily, the second reflects only power and the third reflects only homophily. This interpretation and the results that follow are in line with the findings of van der Leij and Goyal (2006) regarding the core periphery architecture of economists' coauthorships, in which strong ties are found to exist mainly between core members.

[17] Following Boorman and Levitt (1973) one can interpret "power based" society in a genetic context. Every individual would like to establish a link with a bearer of better genes in order to increase his siblings' fitness. However, it is hard to apply it to indirect connections.

networks can be characterized by the pattern of the direct connections between the core agents and the periphery agents (see figure 2). For every periphery member, $i \in L$, define his core as the set $M_i = \{j \mid j \in K, ij \in g\}$ and denote its size by $m_i = |M_i|$. For every core member, $j \in K$, define his periphery as the set $N_j = \{i \mid i \in K, ij \in g\}$ and denote its size by $n_j = |N_j|$ (denote the size of the biggest periphery by $\overline{N} = \max_j n_j$ and the size of the smallest periphery by $\underline{N} = \min_j n_j$). A core-periphery network g is *disconnected* if there are no direct connections between periphery agents and core agents ($\forall i \in L : m_i = 0$). A core-periphery network g is *maximally connected* if each periphery agent is directly connected to all core agents ($\forall i \in L : m_i = |K|$). Note that if the division of the agents to core agents and periphery agents is known, the disconnected core-periphery network and the maximally connected core-periphery network are unique. A core-periphery network g is *minimally connected* if each periphery agent is directly connected to exactly one core agent ($\forall i \in L : m_i = 1$). A minimally connected core-periphery network g is *one-gate* if all periphery agents are directly connected to the same core agent (the gate) and only to her ($\forall i \in L : m_i = 1$ and $\forall i, j \in L : M_i = M_j$)[18]. If the division of agents to subsets is known, then the one-gate minimally connected core-periphery network is unique under the unlabeled set of networks (similar to the star network).

[18] For the importance of the exact characterization of the links between heterogeneous groups see discussion in page 96 of Zeggelink (1995) and especially footnote 11.

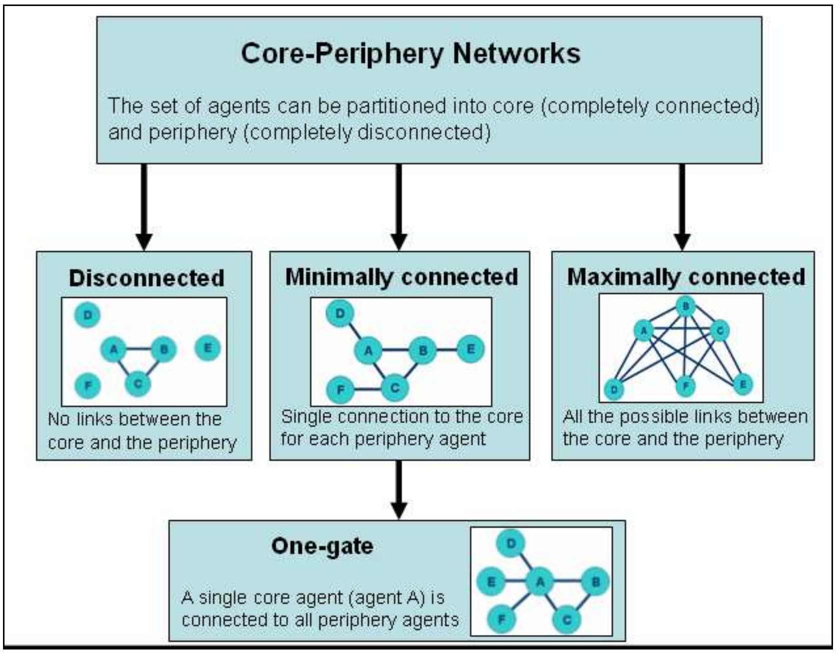

Figure 2: Core-periphery networks (agents A,B,C are the core agents, agents D,E,F are the periphery agents).

3 Results

Helpful lemma

Define the *relative contribution* of $k \in N(i,g)$ to the connection between agent *i* and agent *j* in *g* by $RC(i,j,k,g) \equiv \frac{s_{ij}^{k}(2)}{s_{ij}}\delta^{d_{ij}} f(t_i,t_j)$[19]. It is therefore trivial to note that $\sum_{k \in N(i,g)} RC(i,j,k,g) = \delta^{d_{ij}} f(t_i,t_j)$. Define the *total relative contribution* of neighbor *k* by $TRC(i,k,g) \equiv \sum_{j \in \tilde{N}(i,g)} RC(i,j,k,g) - c$. Note that $TRC(i,k,g) \neq TRC(k,i,g)$ and that $\sum_{k \in N(i,g)} TRC(i,k,g) = u_i(g)$.

Lemma 1: If *g* is pairwise stable then for each pair of agents *i* and *k*, such that $ik \in g$, $TRC(i,k,g) \geq 0$.

[19] This notion of contribution is intuitively close to the betweeness centrality measure (see Freeman 1982).

The proof (as all other proofs) is relegated to the appendix. We will use this lemma in some of the following proofs. One implication of this lemma is that since $u_i(g) = \sum_{k \in N(i,g)} TRC(i,k,g)$, if g is a pairwise stable network, then all the agents have non-negative utility[20].

Extremely low linking costs

Proposition 1 shows that when the linking costs are extremely low, the complete network will emerge both as the predicted outcome and as the favorable outcome. This result is very common in network formation models with positive externalities and it is independent of the preferences of both types of agents (the ordering of the values of the intrinsic value function). We might interpret this result as showing that when the linking costs are very low, the social structure does not reflect the social heterogeneity.

Proposition 1: If $(\delta - \delta^2) w_{\min} > c$ the complete network is the unique pairwise stable network and the unique efficient network.

Low linking costs

These costs are high enough for a direct connection between type *b* agents not to be worthwhile if the pair have an alternative length two path between them. However, these costs are low enough for a direct connection between a type *b* agent and a type *a* agent to be worthwhile even if they have a length two path between them. Proposition 2 shows that, in this case, both the predicted and the socially favorable outcome is the maximally connected core-periphery network in which type *b* agents drop their internal direct connections. Thus, the strength of type *a* agents is reflected in their social position, since they are both highly connected and serve as bridges for the type *b* agents.

[20] In the homogeneous connections model of Jackson and Wolinsky (1996), this implication can extend the results stated above since it establishes that empty network is the unique pairwise stable network in the extremely high costs range ($\delta + \frac{n-2}{2}\delta^2 < c$). If there is another pairwise stable network in this range, its total value should be non-negative since each of the agents have non-negative utility. However, the empty network is the unique efficient network, meaning, there is no other network with non-negative total utility – contradiction.

Proposition 2: If $\left(\delta-\delta^2\right)w_2 > c > \left(\delta-\delta^2\right)w_3$ the maximally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is the unique pairwise stable network and the unique efficient network.

Additional assumptions

To analyze the probable and favorable network structures when the linking costs are higher than $\left(\delta-\delta^2\right)w_2$ we will add one assumption regarding the preferences of the type *a* agents and one assumption regarding the preferences of the type *b* agents.
To demonstrate the need for these additional assumptions assume that agent *i* have a shortest path of length *l>1* both to a type *a* agent and to a type *b* agent. Moreover, assume that shortening the path to these agents does not shorten any other connection that agent *i* possesses. The preferences of agent *i* suggest that she will prefer to form a direct link with the type *a* agent over forming a direct link with the type *b* agent. However, if initially her path to the type *b* agent was longer than her path to the type *a* agent, her preferences regarding the direct links formation are unclear. The two new assumptions are introduced in order to extend the description of the agent's preferences to include some of these cases.

Assumption 1: $\left(\delta-\delta^2\right)w_1 > \delta w_2$.
Assumption 2: $\left(\delta-\delta^2\right)w_2 > \left(\delta-\delta^3\right)w_3$.

The first assumption states that type *a* agent prefers to connect directly to another type *a* agent to whom she otherwise has a path of length two over connecting directly to a type *b* agent to whom she otherwise has no path at all. The second assumption is somewhat weaker and it states that type *b* agent prefers to connect directly to type *a* agent to whom she otherwise has a path of length two over connecting directly to a type *b* agent to whom she otherwise has a path of length three[21]. It is important to note

[21] Another interpretation of the first assumption can be seen if the inequality is written as $\delta w_1 > \sum_{i=1}^{\infty}\delta^i w_2$. Thus, a type *a* agent prefers to connect directly to a type *a* agent to whom she otherwise have no path at all over connecting directly to a type *b* agent which is positioned at the beginning of an infinite line of type *b* agents to none of whom she otherwise has any path at all. Similar

that these interpretations to the assumptions refer only to situations in which forming the link does not yield any shortening of paths to agents other than the agent with whom the link is formed.

Mathematically, these assumptions restrict the eligible values for the intrinsic values function and for the depreciation rate parameter, beyond the previous restrictions ($w_1 > w_2 > w_3$ and $0 < \delta < 1$). One approach is to interpret the assumptions as an introduction of an effective upper bound to the depreciation rate parameter. It is trivial to see that for both assumptions to hold simultaneously, the depreciation rate parameter should satisfy $0 < \delta < \min\left\{1 - \frac{w_2}{w_1}, \frac{w_2}{w_3} - 1\right\}$[22]. Another approach is to interpret these assumptions as a construction of lower bounds to the cardinal difference (or ratio) between the agents' utility from a direct connection with a type *a* agent and her utility from a direct connection with a type *b* agent. The small lower bound set for type *b* agents compared to the one set for type *a* agents, might be interpreted as an addition of a second-order homophily effect. Under this interpretation, type *a* agents are attracted to other type *a* agents both because of their exogenous power and their mutual similarity. However, type *b* agents are attracted to type *a* agents despite the offsetting effect of their differences.

Stronger version of assumption 2, which is symmetric to assumption 1, states that type *b* agent prefers to connect directly to type *a* agent to whom she otherwise has a path of length two over connecting directly to a type *b* agent to whom she otherwise has no path at all.

Assumption 2*: $(\delta - \delta^2)w_2 > \delta w_3$.

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interpretation to the second assumption arises from writing the inequality as $\delta w_2 > (\delta + \delta^2)w_3$. Thus, a type *b* agent prefers to connect directly to a type *a* agent to whom she otherwise has no path at all over connecting directly to a connected pair of type *b* agents to whom she otherwise has no path.

[22] Given w_1 and w_3, the effective restriction is the second assumption iff $w_2 < \frac{2w_1 w_3}{w_1 + w_3}$. An increase in w_2 causes with type *a* agents to be relatively less attractive for type *a* agents and relatively more attractive for type *b* agents. Therefore, given w_1 and w_3, an increase in w_2 turns the first assumption to be the effective restriction. Note that the upper bound can be almost as low as zero (if either $w_2 = w_1$ or $w_2 = w_3$ is approached) and as high as $\frac{w_1 - w_3}{w_1 + w_3}$ (if $w_2 = \frac{2w_1 w_3}{w_1 + w_3}$) which is strictly lower than unity.

It is again trivial to see that for assumptions 1 and 2* to hold simultaneously, the depreciation rate should satisfy $0<\delta<\min\left\{1-\frac{w_2}{w_1},1-\frac{w_3}{w_2}\right\}\le\min\left\{1-\frac{w_2}{w_1},\frac{w_2}{w_3}-1\right\}$[23].

Medium linking costs

Proposition 3 analyses the linking costs range in which a direct connection between type *a* and a type *b* agents is not worthwhile if they have a length two path between them, but it is worthwhile if this link is the only path between them. This proposition asserts that under the assumptions above, the socially favorable outcome is the one gate minimally connected core periphery network in which the core contains all the type *a* agents while the periphery contains all the type *b* agents. However, the set of possible networks is much larger and includes two structures of networks. First, all the minimally connected core periphery networks in which the core contains all the type *a* agents while the periphery contains all the type *b* agents (therefore the favorable networks are also possible). Second, some of the connected networks in which the type *a* agents form a complete clique and there is at least one type *b* agent who is not connected directly to a type *a* agent.

Proposition 3: If $\delta w_2>c>\left(\delta-\delta^2\right)w_2$ and assumption 1 and 2 hold:

1. Every minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is pairwise stable.
2. The set of pairwise stable networks includes also some connected networks in which all type *a* agents are directly connected to each other and there is at least one type *b* agent who is not directly linked to any type *a* agent.
3. The one-gate minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is uniquely efficient.

[23] Given w_1 and w_3, the effective restriction is the second assumption iff when $w_2<\sqrt{w_1w_3}$. Note that $\sqrt{w_1w_3}>\frac{2w_1w_3}{w_1+w_3}$ and therefore the interval of values for which the effective restriction is the second assumption is wider. Note that the upper bound can be almost as low as low as zero (if either $w_2=w_1$ or $w_2=w_3$ is approached) and as high as $1-\frac{\sqrt{w_3}}{\sqrt{w_1}}$ (if $w_2=\sqrt{w_1w_3}$).

The first part of the proof establishes that in the medium linking costs range, the behavior of agents of any minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* follows the following rules:

- No pair of core agents wants to severe their link due to assumption 1.
- No pair of periphery agents likes to form a link due to assumption 2.
- Core agents maintain their links with their own periphery agents since otherwise they will have no access to them.
- Core agents do not form a link to other periphery agents both since they can access them through other core agents and since they do not provide any additional value.

In this architecture, type *a* agents consider other type *a* agents attractive for two reasons - the high intrinsic value of their connection and the access to their periphery agents. As the size of the periphery of the type *a* agent decreases he becomes less attractive to his fellow type *a* agents. Assumption 1 guarantees that even the least valuable type *a* agent, one who has no periphery agents of his own, will still be attractive to other type *a* agents[24].

A pair of periphery agents either shares the same core agent or not. If they share a core agent the value of their connection is $\delta^2 w_3$ while if they have different core agents the value is only $\delta^3 w_3$. Obviously, the later pair has stronger incentive to form a direct link. Hence, we need assumption 2 to ensure that a pair of type *b* agents that have different core agents does not wish to form a direct link. Note, however, that one-gate minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b*, will remain pairwise stable even if assumption 2 is dropped since all the periphery agents in this network share the same core agent.

The second part of the proof characterizes the non core periphery pairwise stable networks as connected networks in which all type *a* agents are directly connected and there is at least one type *b* agent who is not directly linked to any type *a* agent. Under

[24] Weakening the first assumption to $(\delta - \delta^2)w_1 + \left\lfloor \frac{l}{k} \right\rfloor (\delta^2 - \delta^3) w_2 > \delta w_2$ narrows the set of pairwise stable core periphery networks to be the set of minimally connected core periphery networks in which the core includes only type a agents, the periphery includes only type b agents and $\underline{N} = \left\lfloor \frac{l}{k} \right\rfloor$. Note that this discussion is relevant only for $k \geq 3$. When there are one or two type *a* agents assumption 1 is not needed.

certain conditions, that satisfy the costs range and assumptions 1 and 2 (but not 2*), the non core periphery network in figure 3A is pairwise stable. If assumption 2 is replaced by the stricter assumption 2*, we can further establish that the type *b* agents who are not directly linked to any type *a* agent have to posses at least two links. Under certain conditions, that satisfy the costs range and assumptions 1 and 2* (and therefore also 2), the non core periphery network in figure 3B is pairwise stable.

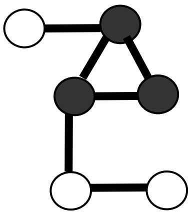

Figure 3A: non core periphery network which is pairwise stable under certain conditions that satisfy assumptions 1 and 2 (not 2*). (black – type *a*, white – type *b*).

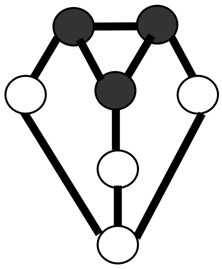

Figure 3B: non core periphery network which is pairwise stable under certain conditions that satisfy assumptions 1 and 2* (and therefore also 2). (black – type *a*, white – type *b*).

T[illegible] re-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is uniquely efficient. The short distance between the type *b* agents provides the intuition for the efficiency of the one-gate network in comparison to other minimally connected core periphery networks. The one gate network could be considered as a mixture of a complete network of the type *a* agents and a star network of the type *b* agents (centered by a type *a* agent). The efficiency of this mixture is not surprising considering proposition 1 of Jackson and Wolinsky (1996). Note that this part of the proof does not use assumption 2 (assumption 1 is needed for the efficiency of the type *a* agents' organization). However, the one gate network is not efficient in the Paretian sense since the gate agent is better off in any other minimally connected core periphery network, since a direct connection between type *a* and a type *b* agents is not worthwhile if they have a length two path between them.

Proposition 3 exhibits the first case of tension between probable and favorable networks. Although this tension can be mitigated by a central planner, since the favorable network is also probable, it demonstrates clearly two distinct sources of inefficiency. One source of inefficiency is non optimal positioning, meaning that some agents have "wrong" friends. The other source of inefficiency is non optimal connectivity, meaning that some agents have "too many" friends. The first source is demonstrated by the stable and inefficient minimally connected core periphery networks. In these networks the inefficiency is a result of lack of coordination between agents in designating the gate agent. The second source of inefficiency is best demonstrated by the set of non core periphery networks under assumptions 1 and 2*. First note that efficient network has $\frac{k(k-1)}{2}$ internal core links and l links in which the type b agents are involved (their links to the gate). The number of links in the non core periphery stable networks is strictly higher since they have the same number of internal core links but more than l links in which type b agents are involved because agents who are not connected to the core cannot be "loose ends". Thus, in these stable networks another source of inefficiency is non optimal connectivity, the agents are over-connected[25].

[25] Under assumption 2, not all of these networks are over connected, see figure 3A.

Additional definition

An additional definition is needed before analyzing the structures emerging in environments with higher levels of linking costs ($c > \delta w_2$). Let *g* be a one gate minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* and let g' be the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b*. Note that the number of links in *g* is the number of links in g' plus the number of type *b* agents (*l*). Thus, the number of additional payments for direct connections in *g*, relative to g' is 2*l*. Denote by *Q* the *additional utility from g per additional payment* -

$$Q = \delta w_2 + \delta^2 w_2 (k-1) + \delta^2 w_3 \frac{(l-1)}{2} - c .$$

Intuitively, *Q* is the net social return from connecting all type *b* agents into the central component of the network. If $Q > 0$ it is beneficial for the whole society to incorporate the weak agents into the central component and otherwise it is not[26]. This social consideration is not in direct accordance with the individual preferences of the agents over the formation of these links. Therefore, *Q* will serve as useful methodological tool in analyzing the tension between stability and efficiency that will arise in the following results. Moreover, *Q* is an increasing function of the network size and therefore this characteristic of the network, which had almost no role in the lower linking costs, is expected to have a direct effect on the range of linking costs in which the stability-efficiency tension exists[27].

High linking costs

Proposition 4 analyses the linking costs range in which a direct connection between type *a* and a type *b* agents is not worthwhile even if this link is the only path between them while a direct connection between a pair of type *a* agents is worthwhile even if otherwise they have a path of length two between them. Assumption 1 guarantees that this range exists. It should also be noted that if $n \geq 3$ this range surely contains an interval in which $Q > 0$ but it may also contain a higher interval in which $Q < 0$.

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[26] This consideration was irrelevant for smaller linking costs since $Q > \delta w_2 - c$ and therefore always positive. Thus, it was always beneficial to incorporate the weak agents into the society.

[27] The size of the network had similar effect in the high linking costs range of the homogeneous model. See propositions 1.2 and 2.4 in Jackson and Wolinsky (1996).

The proposition shows that the characterization of the favorable and probable networks depend heavily on the value of *Q*. If $Q < 0$ (proposition 4.1) the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is the unique pairwise stable network and the unique efficient network. Thus, when $Q < 0$ there is no tension between stability and efficiency. However, if $Q > 0$ (proposition 4.2) the tension exists and it cannot be mitigated by a central planner. The socially favorable outcome in this case is, as in the medium linking costs range, the one gate minimally connected core periphery network in which the core contains all the type *a* agents while the periphery contains all the type *b* agents. The set of probable networks, on the other hand, includes the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* and some other, non core-periphery networks. This result, that the favorable network is not probable, resembles the one found by Jackson and Wolinsky (1996) in the homogeneous model for the range $\delta < c < \delta + \frac{n-2}{2}\delta^2$.

Proposition 4.1: If $(\delta - \delta^2)w_1 > c > \delta w_2$, $Q < 0$ and assumption 1 holds, the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is the unique pairwise stable network and the unique efficient network.

Proposition 4.2: If $(\delta - \delta^2)w_1 > c > \delta w_2$, $Q > 0$ and assumption 1 holds:

1. The disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is pairwise stable.
2. The other members of the set of pairwise stable networks are non core periphery networks in which all type *a* agents are directly connected to each other and every type *b* agent is either isolated or possesses at least two links[28].
3. The one-gate minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is uniquely efficient.

[28] A conjecture we fail to prove or refute is that if assumption 2 holds then the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is the unique pairwise stable network. Note that besides this conjecture, assumption 2 is unneeded to get the results stated in proposition 4.

Proposition 4 is divided only for presentational convenience, the proofs of these propositions are combined and relegated to the appendix.

The first part of the proof shows that when the linking costs are high, type *a* agents in the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b*, do not wish to severe their links with each other but are not willing to form links with the completely isolated type *b* agents. The reluctance of type *a* agents to form links with type *b* agents does not stem only from the low value they give to this kind of direct connection, but also from the fact that the type *b* agents do not provide any "extra" value of short paths to third parties. However, this additional requirement of "extra" value is not demanded when a link between two type *a* agents is considered, since this link bears a high enough intrinsic value to overcome the linking costs.

The second part of the proof shows the efficiency of the disconnected core-periphery network (proposition 4.1) and the one-gate core periphery network (proposition 4.2). The proof technique is very similar to the efficiency proof of proposition 3. However, the differences are due to the behavior of the type *a* agents who no longer wish to form links with isolated type b agents, and therefore the possibility of an efficient network which is not connected. The final step of this part was to use Q to characterize the cases in which the disconnected core-periphery network has higher total utility than the one-gate core periphery network and vice versa.

The third part of the proof completes proposition 4.1 by establishing the uniqueness of the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b*. Briefly, we show that every pairwise stable network have to contain a complete clique of all type *a* agents. Thus, any other pairwise stable network must have additional links relative to the disconnected core-periphery network. By lemma 1 its total utility should be higher, contradicting the efficiency result.

The last part of the proof characterize, for the case of $Q > 0$, the set of pairwise stable networks other than the disconnected core periphery network. It is easily shown that these networks are non core periphery networks in which all type *a* agents are directly connected to each other and every type *b* agent is either isolated or possesses at least two links. We show that under values that do not satisfy assumption 2, the first case can be demonstrated by a network with two connected type *a* agents and a separate circle of eleven type *b* agents while the second case can be demonstrated by network pictured in figure 4.

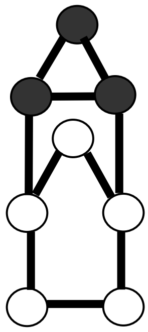

Figure 4: non core periphery network which is pairwise stable under certain conditions that do not satisfy assumption 2. Every type b agent has at least two links (black – type *a*, white – type *b*).

The main characteristic of this range is that the linking costs are too high for any type *a* agent to invest in a connection with an otherwise isolated type *b* agent. Obviously, if this investment is too high for a type *a* agent, it is also too high for a type *b* agent. Therefore, all the probable networks have either isolated type *b* agents or type *b* agents that are directly connected to at least two different agents.

The additional utility per additional payment, denoted by *Q*, is a general measure that has an interesting role in the results of proposition 4. While it is trivial, by definition, that its sign sets the efficient network, it is rather surprising that its sign also have some relation to individual incentives since it distinguish between parameter values for which there are pairwise stable non disconnected core periphery networks and cases in which the disconnected core periphery network is the unique stable network.

As a result, Q serves as indicator to the tension between favorable and stable networks which exists only if $Q > 0$. However, the tension in this case is worth than in all previous cases since the favorable one gate core periphery network is not stable since its type *b* agents are not attractive enough for the potential gate agent. As mentioned above, this result resembles the one found by Jackson and Wolinsky (1996) in the homogeneous model for stars in the range $\delta < c < \delta + \frac{n-2}{2}\delta^2$.

Extremely high linking costs

Proposition 5 analyses the linking costs range in which a direct connection between a pair of type *a* agents is not worthwhile if they have an alternative path of length two between them. The proposition shows that in this range, if assumption 1 holds, core periphery networks are neither pairwise stable nor efficient. The basic intuition behind this result is that in this linking costs range, the clique architecture is too costly for the type *a* agents. Indeed, although it would not be proven here due to lack of interest, various architectures that feature a star for the type *a* agents emerge as pairwise stable and efficient in this range of linking costs.

Proposition 5: If $c > (\delta - \delta^2)w_1$, $k \geq 3$ and assumption 1 holds, no core periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is either pairwise stable or efficient.

The instability result is fairly obvious. Already, in proposition 4 we saw that if the linking costs are too high for a type *a* agent to connect to an otherwise isolated type *b* agent, the disconnected core periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is the only core periphery candidate for pairwise stability. In the present level of linking costs, this network is not stable since it is too high for the a type *a* agent to maintain connections to all the other agents considering he has an alternative path of length two to each of them if $k \geq 3$.
Note that assumption 1 is crucial for the correctness of this part of the proposition. If it does not hold and $c > \delta w_2$, pairwise stable core periphery networks might emerge (numerical example is provided in the proof). The intuition is that the linking costs do not prevent type *a* agents from connecting to an otherwise isolated type *b* agents. By maintaining this kind of connections the type *a* agents become more attractive to other type *a* agents since a link with them provides additional shorter paths to their peripheral type *b* agents. If all the type *a* agents increase their attractiveness by connecting to otherwise isolated type *b* agents, it might be worthwhile for all the internal connections in the core to be kept and stability to be achieved.
In order to show that core periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* are inefficient, we began by showing that an efficient core-periphery network should minimize the paths between peripheral type *b* agents. Then any network in which there was a type *b* agent with more than one link was shown to be inefficient. The last step was to show that the rest of the core periphery networks, where all the non isolated periphery agents are linked to the same type *a* agent (the gate) are inefficient. Indeed, a non core periphery architecture in which all the type a agents and the non isolated type *b* agents are organized as a star around the gate is shown to have higher total utility. Note that assumption 1 was not needed for this part of the proof.

4 Conclusions

In this paper we introduced the heterogenuous connections model in which there are two types of agents whose their benefit from their connections to other agents depends on their geodesic distance and on their types. The dependence of the benefit on the types is modeled using a discrete, positive and symmetric intrinsic value function that multiply the original depreciation factor of the homogeneous connections model of Jackson and Wolinsky (1996).

In the case analyzed here both types have the same ordinal preferences over connections, holding the path length constant and provided that no indirect benefits are incurred due to shortening paths to other agents. This setting is interpreted as a "power based" society, where the powerful type is the type preferred by both agents.

We show that in this simple framework, the dominant architecture when linking costs are not too low and not too high is the core periphery architecture where the powerful type agents are positioned in the completely connected core while the other type is peripheral and is completely disconnected internally. Various versions of this architecture appear as pairwise stable networks (in some cases the unique stable network) and as efficient networks (always unique).

We suggest heterogeneity and "power indicating preferences" as an alternative explanation for the circumstances under which a core periphery network might emerge. Thus, after the formation of the network, the core agents have two distinct sources of power. The first source is the high intrinsic value that all the members of the society have from connecting to them. This power is exogenous and independent of the network formation process. The second source of power is the central position of the preferred type in the social network. This secondary power which is easily observed through the network structure is both the manifestation of the original power and its perpetuator.

The framework used in this paper, and specifically the intrinsic value function enables the analysis of two other types of heterogenuous social preferences. While we assumed $w_1 > w_2 > w_3$ to characterize the "power based" society, assuming that $w_1, w_3 > w_2$ might be interpreted as a "homophilic" society in which both types of agents prefer to connect to their own type over connecting to the other type. Assuming $w_2 > w_1, w_3$ might be interpreted as a "heterophilic" society in which both types of agents prefer to connect to the other type over connecting to their own type.

It is intuitive to predict that the dominant structure in the "homophilic" society is the segregated network in which there are two cohesive groups densely connected internally, one of type *a* agents and the other of type *b* agents[29]. However, it seems that it takes very high linking costs in order to achieve complete segregation, meaning that the two groups form two separate components, since the benefit from a connection between these groups is huge[30]. This result is consistent, obviously, with Burt (1992) identification of structural holes and the massive gains that they carry. These basic intuitions are backed by an overwhelming amount of empirical evidence that was gathered regarding the dominance of segregated networks in which each component is internally homogeneous under homophilic social preferences[31].

It is also intuitive to predict that the dominant structure in the "heterophilic" society is the bipartite network which consists of two cohesive groups sparsely connected internally and densely connected externally[32]. One observation is that the lack of internal connections will lead to high average degree in this environment[33]. The main line of research that analyses bipartite structures is the analysis of matching procedures, which is fairly different from the formation literature by its mechanistic approach and the lack of network perspective in the agents' utilities. It seems that the empirical literature regarding the network perspective of bipartite structures and heterophily barely exists[34].

We will conclude with the observation that compared with core periphery networks in the empirical literature, the networks that emerged in this analysis were "too neat". Moreover, the change in the stable architecture during an increase of the linking costs wasn't continuous. However, we believe that this is due to the simplicity of the model

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[29] Informally, it seems that the internal structure of each of the cohesive groups is either a complete network or a star, depending on the linking costs, as predicted in the homogeneous model of Jackson and Wolinsky (1996).

[30] Similar intuition can be deduced from proposition 2 of Jackson and Rogers (2005).

[31] For representative results regarding homophily in social networks see Precker (1952), Gurevitch (1961), Travers and Milgram (1969), White et. al. (1976), Verbrugge (1977), Brieger and Ennis (1979), Frank (1995) and McPherson et. al. (2001). There is also a vast theoretical literature concerning segregation, in particular due to the tendencies towards economic segregation in both the US and Europe since the 1970's (i.e. Miyao (1978), Benabou (1993, 1996a, 1996b), Durlauf (1996)).

[32] Some cases of the coauthor model of Jackson and Wolinsky (1996) generate bipartite networks as an efficient (yet not stable) outcomes. However, these networks are not densely connected between the two sets of agents.

[33] Let agent *i* be a type *a* agent and let agent *j* be a type *b* agent. If they are not directly connected, they will probably have only a path of length three between them since most of *i*'s neighbors will be type *b* agents who are sparsely connected to agent *j*, and the same for agent *j*. Thus, the net gain from direct connection is higher in this framework and therefore a high average degree is predicted.

[34] Krackhardt and Hanson (1993) mention segregation, core-periphery and bipartite structures as undesirable architectures of organizational networks. However they do not discuss the normative causes for the formation of these architectures.

and introducing more complex mathematical objects as non linear linking costs or secondary stochastic formation processes will yield these deviations with no substantial important lessons about the formation behavior of the agents.

Appendix

Proof of lemma 1

We will show that if g is pairwise stable then for each pair of agents i and k, such that $ik \in g$, it must be that $TRC(i,k,g) \geq 0$. Assume that g is pairwise stable and that there is a pair of agents i and k, such that $ik \in g$ and $TRC(i,k,g) < 0$. Let $l \in \tilde{N}(i,g)$. First, if $s_{il}^{k}(2) = 0$ then none of the shortest paths between agent i and agent l in g pass through agent k. Thus, $S(i,l,g) = S(i,l,g-ik)$ and $\forall j \in N(i,g-ik): RC(i,l,j,g-ik) = RC(i,l,j,g)$. Second, if $s_{il}^{k}(2) = s_{il}$ then all the shortest paths between agent i and agent l in g pass through agent k. In network g-ik, agent i and agent l are either connected or disconnected. If they are disconnected then $\forall j \in N(i,g-ik): RC(i,l,j,g-ik) = RC(i,l,j,g) = 0$. If they are connected then $\exists j \in N(i,g-ik): RC(i,l,j,g-ik) > RC(i,l,j,g) = 0$ and there might be other neighbors such that $RC(i,l,j,g-ik) = RC(i,l,j,g) = 0$. Last, if $s_{il} > s_{il}^{k}(2) > 0$ then some of the shortest paths between agent i and agent l pass through agent k and others through other neighbors. Let agent m_1 be one of those neighbors. In g-ik, $s_{il}^{m_1}(2)$ is the same as in g and s_{il} decreases and therefore $RC(i,l,m_1,g-ik) > RC(i,l,m_1,g)$. Let agent m_2 be one of the neighbors through which no shortest path between agent i and agent l go (such agent not necessarily exists). Hence, $RC(i,l,m_2,g-ik) = RC(i,l,m_2,g) = 0$. Thus, when the link between agent i and agent k is severed, the relative contributions of i's other neighbors are non-decreasing and by the definition of total relative contribution, stated above, it is clear that $\forall j \in N(i,g-ik): TRC(i,j,g-ik) \geq TRC(i,j,g)$. Thus, $\sum_{j \in N(i,g)\backslash k} TRC(i,j,g) \leq \sum_{j \in N(i,g-ik)} TRC(i,j,g-ik)$. Since $TRC(i,k,g) < 0$ we get that $\sum_{j \in N(i,g)} TRC(i,j,g) < \sum_{j \in N(i,g-ik)} TRC(i,j,g-ik)$. However, this means that $u_i(g) < u_i(g-ik)$. Therefore, it is beneficial for agent i to drop his link to agent k and therefore g is not pairwise stable. Contradiction. Thus, if g is pairwise stable then for each pair of agents i and k, such that $ik \in g$, $TRC(i,k,g) \geq 0$.

Proof of proposition 1

To show that the complete network is pairwise stable first consider the case of $n > 2$. The net gains of agent i from keeping the direct connection ij are at least $\delta w_{\min} - c - \delta^2 w_{\min}$. Since $(\delta - \delta^2) w_{\min} > c$, the net gains are positive and both agents will keep the link. In the case of $n = 2$, both agents are completely isolated after severing ij and therefore their net gains from keeping the link are at least $\delta w_{\min} - c$. Since $\delta w_{\min} > c$, the net gains are again positive and both agents will keep the link. Thus, the complete network is pairwise stable. To prove that the complete network is the unique stable network, assume that there is another stable network, g'. There is at least one pair of agents that are not directly linked in g'. Assume that a path of length two links them. As shown above, their net gains from connecting directly, whatever are their types, are positive and therefore this network is not stable. Obviously, if a longer path links them (as in the case of $n = 2$), they will also prefer to connect directly. Thus, the unique stable network is the complete network. To prove that the complete network is the unique efficient network, first consider the case of $n > 2$ and let g' be a non-complete network. There exists in g' a pair of agents i and j which are not directly connected. Consider the network $g'' = g' + ij$. The minimal difference in total utility between the two networks is achieved when every other agent $h \neq i, j$ have the same utility in g'' as in g'[35], the two agents are only two links away in g' and when the internal value of their connection is the lowest possible one. Thus, the minimal difference in total utility is $2(\delta w_{\min} - c - \delta^2 w_{\min})$. As shown above, this difference is positive and therefore g' is not efficient. If $n = 2$ the only non-complete network is the empty network, in which the total utility is zero. Thus, the difference in total utility between the complete network and the empty network is at least $2(\delta w_{\min} - c)$ and as showed above this difference is positive. In conclusion, for any non-complete network g' in which agents *i* and *j* are not directly connected, the network $g'' = g' + ij$ has higher total utility. Thus, the complete network achieves the highest total utility and therefore it is strongly and uniquely efficient.

<u>Proof of proposition 2</u>

Let *g* be the "maximally connected" core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b*. We will prove that *g* is pairwise stable by showing that no pair of periphery agents wishes to form a direct link, no core agent wishes to severe her direct links to either the core or the periphery agents and no periphery agent wishes to severe her direct links to the core agents. In order to show that no pair of periphery agents wish to form a direct link, note that these agents are of type *b* and that their utility from *g* is $k(\delta w_2 - c) + (l-1)\delta^2 w_3$. If there is more than one periphery agent (otherwise this case is irrelevant), the utility of a periphery agent *i* in $g + ij$ where *j* is a periphery agent is $k(\delta w_2 - c) + (l-2)\delta^2 w_3 + (\delta w_3 - c)$. Thus, since $c > (\delta - \delta^2)w_3$, no periphery agent in *g* wishes to form a direct link with another periphery agent. In order to show that no core agent wishes to severe a direct link with another core agent, note that these agents are of type *a* and that their utility in *g* is $(k-1)(\delta w_1 - c) + l(\delta w_2 - c)$. If there are more than one core agent (otherwise this case is irrelevant), the utility of a core agent *i* in $g - ij$ where *j* is a core agent is $(k-2)(\delta w_1 - c) + \delta^2 w_1 + l(\delta w_2 - c)$ as they will have a path of length two through a third party (either core agent or periphery agent). Thus, the link would be kept since $(\delta - \delta^2)w_1 > c$. In order to show that no core agent wishes to severe a direct link with a periphery agent consider first the case in which there is more than one core agent. If a core agent *i* decides to sever a direct link with a periphery agent *j*, it has no effect on the length of her paths to the rest of the agents in $g - ij$. The new path to agent *j* will be of length two (through another core agent). The utility of the core agent in $g - ij$ is $(k-1)(\delta w_1 - c) + (l-1)(\delta w_2 - c) + \delta^2 w_2$ and she will keep the link since $(\delta - \delta^2)w_2 > c$. If there is only one core agent, her utility from *g* is $l(\delta w_2 - c)$ while her utility from $g - ij$ is $(l-1)(\delta w_2 - c)$ and she will keep the link since $\delta w_2 > c$. In order to show that no periphery agent wishes to severe a direct link with a core agent consider first the case in which there are more than one core agent. If a periphery agent *i* decides to sever a direct link with a core agent *j*, it has no effect on the length of her paths to the rest of the agents in $g - ij$. The new path to agent *j* will be of length two since (through another core agent). The utility of the periphery agent from $g - ij$ is $(k-1)(\delta w_2 - c) + \delta^2 w_2 + (l-1)\delta^2 w_3$ and she will keep the link since $(\delta - \delta^2)w_2 > c$. If there is only one core agent, the utility of a periphery agent from *g* is $(\delta w_2 - c) + (l-1)\delta^2 w_3$ while her utility from $g - ij$ is zero. Since $\delta w_2 + (l-1)\delta^2 w_3 \geq \delta w_2 > c$, she will keep the link. In conclusion, we showed that *g*, the maximally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is pairwise stable. To prove that this

[35] Note that in the connections model the externality of two players connecting on the other members of the network is non-negative. The new link might not change any shortest paths in the network (except of the one between the two connecting agents) or replace certain paths by shorter paths. In both cases, the utility of the members of the network, apart from the two that establish the new link, is non-decreasing. Deleting a link, on the other hand, might harm agents that are not involved directly in the severed link since it might lengthen some of their shortest paths. Thus, if two agents wish to add a link it will surely increase the total utility of the network, while if an agent whishes to severe a link it improves her utility but might harm total utility.

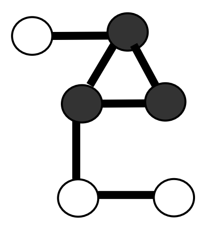
network is the unique pairwise stable network we will first show that in any pairwise stable network all the pairs of type *a* agents are directly connected. Let g' be a pairwise stable network in which there is a pair of type *a* agents who are not directly connected (if there is only one type *a* agent this case is irrelevant). Their minimal gain from linking is achieved if a path of length two links them and if this direct link does not shorten any of their paths to other agents. Thus, their gains from the direct link are at least $\delta w_1 - c - \delta^2 w_1 > 0$. Obviously, if a longer path links them and/or this link shortens their paths to other agents, they will surely gain even more from a direct link and therefore g' is not stable, contradiction. Now we will show that in any pairwise stable network all the pairs of type *a* agent and type *b* agent are directly connected. Let g'' be a pairwise stable network in which there is a pair of type *a* agent and type *b* agent who are not directly connected while all the pairs of type *a* agents are directly connected. Their minimal gain from linking is achieved if a path of length two links them and if this direct link does not shorten any of their paths to other agents. Thus, their gains from this direct link are at least $\delta w_2 - c - \delta^2 w_2 > 0$. Obviously, if a longer path links them and/or this link shortens their paths to other agents, they will gain even more from a direct link and therefore g'' is not pairwise stable, contradiction. So far we have shown that every pairwise stable network has at least all the edges of the maximally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b*. If there is only one type *b* agent there are no other networks with these edges (in fact it is the complete network) and it is the unique pairwise stable network. If there is more than one type *b* agent, we will show that every network which has these links but also some more links between type *b* agents is not pairwise stable. Let g''' be a stable network in which every pair of type *a* agents are directly connected and every pair of type *a* agent and type *b* agent are directly connected and there is at least one pair of type *b* agents which are directly connected. In g''' the path length between two type *b* agents is two if they are not directly connected or one if they are directly connected. Moreover, severing a direct link between two type *b* agents *i* and *j* will not affect the paths between those two agents and other agents in the network. Thus, the net utility gains of each type *b* agent's utility from severing the direct link to another type *b* agent are $\delta^2 w_3 - (\delta w_3 - c)$. Since $c > (\delta - \delta^2) w_3$ she would wish to severe her direct link to the other type *b* agent and therefore g''' is not pairwise stable, contradiction. Therefore, the maximally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is the unique pairwise stable network. To prove that this network is the unique efficient network, let g be a network in which there exist a pair of agents *i* and *j*, at least one of them is a type *a* agent, which are not linked. Consider the network $g' = g + ij$. Remember that the externality of two players connecting on the other members of the network is non-negative and therefore if both agents *i* and *j* wish to link directly to each other the total utility of g' must be higher than the total utility of *g*. We showed above that two type *a* agents always wish to connect directly and so do a pair of type *a* agent and type *b* agent. Thus, the efficient network belongs to the set of networks in which type *a* agents are completely connected while type *b* agents are connected to all type *a* agents and maybe to some of the other type *b* agents. In these networks, the shortest path from a type *a* agent to any other agent is of length one, and the shortest path between two type *b* agents is one if they are directly connected and two otherwise. Thus, severing a link between two type *b* agents harms the utility of none of the agents that are not involved in the link. Since $(\delta - \delta^2) w_3 < c$ any pair of type *b* agents increase total utility by severing the link. Hence, the highest utility will be achieved if there will be no links between type *b* agents. Thus, the unique efficient network is the maximally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b*.

Proof of proposition 3

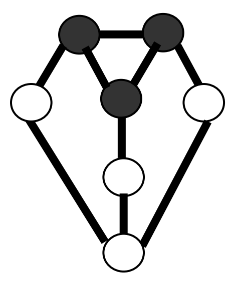
Let g be a member of the set of minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b*. In order to show that *g* is pairwise stable we have to verify four conditions: no pair of periphery agents would like to connect, no pair of core agent and periphery agent which is connected to another core agent would like to connect, no pair of core agent and one of her periphery agents would like to severe their direct link and no pair of core agents would like to severe their direct links. First, we will show that no pair of periphery agents would like to form a link (if there is only one periphery agent this case is irrelevant). Note that there are two kinds of pairs of periphery agents – a pair in which both agents are connected to the same core agent and a pair in which the agents are connected to different core agents. Consider the case in which both periphery agents are connected to the same core agent. If these two periphery agents form a link, it reduces the distance between them but does not get them closer to any other agent. By forming this link, each of these agents gains a net utility of $\delta w_3 - c - \delta^2 w_3$. Thus, since $c > (\delta - \delta^2) w_2 > (\delta - \delta^2) w_3$ this pair of agents would not wish to connect directly. Now, consider the case in which the two periphery agents are connected to different core agents (if there is only one core agent this case is irrelevant). If these two periphery agents form a link, it reduces only the distance between them. By forming this link, each of these periphery agents has a net utility of $\delta w_3 - c - \delta^3 w_3$. Thus, using assumption 2, $c > (\delta - \delta^2) w_2 > (\delta - \delta^3) w_3$ and this pair of agents would not wish to connect directly. Second, we will show that no pair of core agent (j_1) and periphery agent (i) who is linked to another core agent (j_2) would like to form a direct link. If *i* links to j_1, *i* shortens her path to j_1 and to her periphery. On the other hand, j_1 only shortens her path to *i*. Since the gains of j_1 from such a direct link are lower (the intrinsic values are positive and symmetric), she decides whether the link ij_1 will form[36]. Since originally j_1 has a path of length two to *i*, she will object as long as $c > (\delta - \delta^2) w_2$. Therefore, no pair of core agent and periphery agent who is linked to another core agent would like to form a direct link. Third, we will show that no pair of core agent and one of her periphery agents would like to sever their mutual link. The core agent will not sever the link since by severing it she loses $\delta w_2 - c$ and $\delta w_2 > c$. The net utility of the periphery agent from this link is at least as high as that of the core agent since she gets all her indirect connections through this link. Therefore, she will keep the link as well. Last, we will show that no pair of core agents would like to sever their mutual link (irrelevant if there is one core agent). If there are only two core agents in the network, *i* and *j*, severing the link between them will turn the network into two disconnected stars. In this case, the net utility gains to agent *i* from deleting the link are $c - (\delta w_1 + \delta^2 N_j w_2)$. Therefore, she will keep the link since $\delta w_1 + \delta^2 N_j w_2 \geq \delta w_1 > \delta w_2 > c$. If there are more than two core agents in the network, agent *i*'s net utility gains from severing the link are $\delta^2 w_1 + \delta^3 N_j w_2 + c - (\delta w_1 + \delta^2 N_j w_2)$ (her direct contact with agent *j* becomes a length two path, and her length two paths to *j*'s periphery become length three paths). Using assumption 1, $(\delta - \delta^2) w_1 + (\delta^2 - \delta^3) N_j w_2 \geq (\delta - \delta^2) w_1 > \delta w_2 > c$, we get that both core agents will keep the direct link. Thus, we showed that any minimally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b*, is pairwise stable.

However, there are networks that are pairwise stable and do not belong to the set of minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b*.

The first example is pairwise stable under assumption 2 and not pairwise stable under assumption 2*. Consider the following network where the black circles stand for type *a* agents and white circles stand for type *b* agents. One can verify that under the values $c = \frac{11}{32}, \delta = \frac{3}{4}, w_1 = 5, w_2 = 1, w_3 = \frac{1}{2}$, which satisfy the range and assumptions 1 and 2 and does not satisfy assumption 2*, this non-core-periphery network is pairwise stable. If the linking costs were higher than δw_3, assumption 2* was satisfied but the network was not pairwise stable since the connection between the two type *b* agents would be dropped by the agent connected to the core.



The second example is pairwise stable under both assumptions 2 and 2*. Consider the following network where the black circles stand for type *a* agents and white circles stand for type *b* agents. One can verify that under the values $c = 1, \delta = \frac{1}{2}, w_1 = 10, w_2 = 3\frac{1}{2}, w_3 = 1\frac{1}{2}$, which satisfy the range and assumptions 1, 2 and 2*, this non-core-periphery network is pairwise stable.



[36] This observation is a specific case of the "principle of least interest" that states that the party least interested in a relationship determine the intensity of interaction (see Waller and Hill (1951)).

In both examples, if the type *b* agent who is not connected to the core, will severe all his links and will form a link with one of the type *a* agents, the resulting minimally connected core periphery network will be pairwise stable. However, it will not Pareto dominate the original network since the type *a* agent with whom the periphery agent formed the link suffers a loss of utility, because in this range of linking costs, a direct connection between type *a* and a type *b* agents is not worthwhile if they have a length two path between them and it does not shorten any of his other paths.

Now, we will characterize the set of pairwise stable networks which are not minimally connected core-periphery networks under assumptions 1 and 2. We will show first, that each pair of type *a* agent are directly connected in *g*. Let g' be a network in which there is a pair of type *a* agents who are not directly connected. In order to examine the minimal contribution of a direct link to these agents' utilities, assume that a path of length two links them and that connecting them directly does not shorten any of their other connections. Their net utility gains from a direct connection are $\delta w_1 - c - \delta^2 w_1$. By assumption 1, $(\delta - \delta^2)w_1 > \delta w_2 > c$ and therefore it will surely be beneficial for these agents to connect directly, let alone if a longer path links them and/or if the new link shortens their connections to other agents. Thus, g' is not pairwise stable, and the set of pairwise stable networks is a subset of the set of all networks in which each pair of type *a* agents is directly connected. Next we will show that a pairwise stable network must be connected. Let g'' be a network in which each pair of type *a* agents are directly connected and there is a pair of agents with no path between them. One component of this network includes at least all the type *a* agents while all the other components include only type b agents. Since $\delta w_2 > c$ it is beneficial for any pair of type a agent and type b agent who do not share the same component to connect, even if they supply each other with no indirect shorter paths to other agents. Thus, g'' is not pairwise stable, and the set of pairwise stable networks is a subset of the set of all connected networks in which each pair of type *a* agents is directly connected. Next, let g''' be a connected network in which each pair of type *a* agents is directly connected, each type *b* agent is directly connected to at least one type *a* agent and there is at least one type *b* agent (agent *i*) which is directly connected to more than one type *a* agent (agents $j_1, j_2, \ldots, j_k$). Note that agent j_1 has a path of length two to agent *i* through agent j_2 and that none of her shortest paths pass through this agent (every type *b* agent is at least directly connected to one type *a* agent). Therefore her net gains from severing its link to agent *i* are $\delta^2 w_2 - (\delta w_2 - c)$. Thus, since $c > (\delta - \delta^2)w_2$, j_1 would like to severe her direct link to agent *i* and g''' is not pairwise stable. In conclusion, a pairwise stable network must be a connected network such that each pair of type *a* agents is directly connected. Moreover, there are two possible patterns of connections between the type *a* and type *b* agents – either each type *b* agent is directly connected to exactly one type *a* agent or there is at least one type *b* agent who is not directly connected to any type *a* agent. Regarding the first pattern, it is left to be shown that in any such pairwise stable network there are no direct connections between type *b* agents. Note that there are two kinds of pairs of type *b* agents – a pair in which both agents are connected to the same type *a* agent and a pair in which the agents are connected to different core agents. Consider the case in which both type *b* agents are connected to the same type *a* agent. Keeping a direct link provides net utility gains of $\delta w_3 - c - \delta^2 w_3$ and since $c > (\delta - \delta^2)w_3$ this pair of agents would prefer to sever the link. Consider the case in which each of the two agents is connected to a different type *a* agent. Keeping the link provides net utility gains of $\delta w_3 - c - \delta^3 w_3$. Using assumption 2, $c > (\delta - \delta^2)w_2 > (\delta - \delta^3)w_3$, ensures that this pair of agents would prefer to sever the link. Thus, if *g* is pairwise stable of the first pattern it must be a minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b*. The pairwise stable networks of the second pattern are connected networks in which each pair of type *a* agents is directly connected and there is at least one type *b* agent who is not directly connected to any type *a* agent. Replacing assumption 2 by assumption 2* forces these type *b* agents who are not directly connected to any type *a* agent to have at least two links. Otherwise, they have one link (the network is connected) and the net utility gain of the agent that they are linked to, from severing this link is $c - \delta w_3$. By assumption 2* this gain is positive and therefore under assumptions 1 and 2* the set of non-core-periphery networks is the set of connected networks in which each pair of type *a* agents is directly connected, there is at least one type *b* agent who is not directly connected to any type *a* agent and each one of these agents has at least two links. The two examples above demonstrate exactly this point.

Now we will prove that the set of efficient networks is the set of one-gate minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b*. Let g be a network in which there exists a pair of agents *i* and *j*, both of them are type *a* agents, which are not linked. Consider the network $g' = g + ij$. The minimal difference in total utility will be the differences in these two agents' utilities assuming that they have a path of length two between them in *g* and that the new link does not improve any other shortest path in the network (see footnote 35). Thus, the minimal difference for both agents is $2(\delta w_1 - c - \delta^2 w_1)$ which is positive due to assumption 1 $(\delta - \delta^2)w_1 > \delta w_2 > c$. Therefore, for any network *g* in which there exists a disconnected pair of agents *i* and *j*, both of them are type *a* agents, there is a network $g' = g + ij$ with higher total utility. Thus, the efficient network belongs to the set of networks in which type *a* agents are completely connected among themselves. Next we will show that the efficient network is a member of the set of connected networks in which type *a* agents are completely connected among themselves. Let g'' be a network in which each pair of type *a* agents are directly connected and there is a pair of agents with no path between them. One component of this network includes at least all the type *a* agents while all the other components include only type b agents. Consider a pair of type *a* agent (*i*) and type *b* agent (*j*) who do not share the same component. Such a pair exists in any disconnected network. The minimal difference in total utility if these agents connect, will be the differences in these two agents' utilities assuming that the new link does not improve any other shortest path in the network (see footnote 35). Thus, the minimal difference for both agents is $2(\delta w_2 - c)$ which is positive since $\delta w_2 > c$. Thus, g'' is not efficient since the total utility in $g'' + ij$ is strictly higher. Thus, the efficient network is the network that maximizes the total utility of connected networks in which each pair of type *a* agents is directly connected. The last step is to show that the one-gate minimally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is this network. In any connected network in which each pair of type *a* agents is directly connected, there are $\frac{k(k-1)}{2}$ paths between two type *a* agents, kl paths between type *a* and type *b* agents and $\frac{l(l-1)}{2}$ paths between two type *b* agents. Since there is a complete sub-graph of the type *a* agents all their internal $\frac{k(k-1)}{2}$ paths are direct links. Denote by $K_1 \geq 1$ the number of direct links between type *a* and type *b* agents (it must be at least one since it is a connected network) and by $K_2 \geq 0$ the number of direct links between two type *b* agents. Thus, there are $kl - K_1$ indirect links between type *a* and type *b* agents and $\frac{l(l-1)}{2} - K_2$ indirect links between two type *b* agents. In addition, since there are *l* type *b* agents and the network is connected, it must be that $K_1 + K_2 \geq l$. The maximal overall value of this network is achieved when all the indirect links are of length two, and it is:

$$\frac{k(k-1)}{2}(2\delta w_1 - 2c) + K_1(2\delta w_2 - 2c) + K_2(2\delta w_3 - 2c) + 2\delta^2 w_2(kl - K_1) + 2\delta^2 w_3\left(\frac{l(l-1)}{2} - K_2\right)$$

The overall value of a one-gate minimally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is:

$$\frac{k(k-1)}{2}(2\delta w_1 - 2c) + l(2\delta w_2 - 2c) + 2\delta^2 w_2(k-1)l + 2\delta^2 w_3 \frac{l(l-1)}{2}.$$

The difference between the maximal value and the one-gate network value is $2(K_1 - l)(\delta w_2 - \delta^2 w_2 - c) + 2K_2(\delta w_3 - \delta^2 w_3 - c)$. Note that this difference has to be non-negative since the maximal value has to be at least as high as the total utility of the one-gate minimally connected network. Therefore, it must be that $(l - K_1)(c - \delta w_2 + \delta^2 w_2) \geq K_2(c - \delta w_3 + \delta^2 w_3)$. Since $c > (\delta - \delta^2)w_2 > (\delta - \delta^2)w_3$, it holds that $c - \delta w_2 + \delta^2 w_2 < c - \delta w_3 + \delta^2 w_3$ and therefore it must be that either $l - K_1 > K_2$ or $K_1 = l$ and $K_2 = 0$. Note that the first option violates the connectivity condition - $K_1 + K_2 \geq l$. In conclusion, the highest total utility among connected networks is achieved if this network has the type *a* agents completely connected among themselves, no connections between type *b* agents and *l* connections between type *a* agents and type *b* agents. Due to connectivity it must be that the *l* links between type *b* agents and type *a* agents are divided such that each type *b* agent has exactly one such link. Therefore, the network belongs to the set of minimally connected core-periphery networks. Moreover, we showed that the one-gate minimally connected core-periphery network achieves the maximum. To show that other minimally connected core-periphery networks do not achieve the maximum note that if all type *b* agents connect to the same type *a* agent it increases the utility from the indirect links within the type *b* agents and does not change the utility from other types of connections (the connections between type *a* agents are still of length one and *l* of the intertype connections are of length one and the rest are of length two). Hence, the network that achieves the highest total utility among all connected networks is the one-gate minimally connected core-periphery networks in which all the core agents are of type *a* and all the periphery agents are of type *b*. Thus, we showed that one-gate minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b* maximizes the total utility of connected networks in which all type *a* agents are completely connected among themselves. Since earlier we showed that disconnect networks and networks in which there are type *a* agents which are not directly connected are inefficient, the one-gate minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b* is strongly efficient and there are no other efficient networks.

Proof of proposition 4

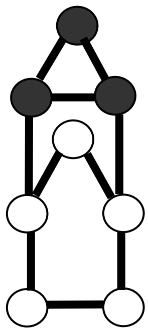
Let *g* be the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b*. To show that *g* is pairwise stable we have to verify that pairs of type *a* agents would not like to severe their link, while any other pair of agents would not like to form a link. First, let us consider the links between the periphery agents. The value to the agents from being completely isolated is zero while the value for each of them from being directly connected is $\delta w_3 - c$. Since $c > \delta w_2 > \delta w_3$ no pair of periphery agents in the disconnected core-periphery network would like to form a link. Second, let us consider a pair of a periphery agent and a core agent. The core agent gains, by forming the link, $\delta w_2 - c$ since no indirect connections are formed through the periphery agent. Since $c > \delta w_2$ no pair of core agent and periphery agent in the disconnected core-periphery network would like to form a link (note that the considerations of the periphery agent are irrelevant in this case due to the mutual consent requirement). Third, let us consider the link between the core agents. If there are only two core agents in the network, severing the link between them will turn the network into the empty network and therefore they keep the link since $\delta w_1 > c$. If there are more than two core agents in the network, a core agent gains the cost of the link from severing the link. In addition, her direct contact with her fellow core agent becomes 2-link path. Thus, her net utility gains from severing the link are $\delta^2 w_1 - (\delta w_1 - c)$ and therefore the agent will keep the link since $(\delta - \delta^2)w_1 > c$. In conclusion, we showed that the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is pairwise stable. Note that this observation does not depend on the value of Q and therefore it is relevant for both 4.1 and 4.2. We will deal with the uniqueness of this pairwise stable network after the proving the efficiency results for both 4.1 and 4.2. We will prove that the efficient network is either a one-gate minimally connected core-periphery network where all the core agents are of type *a* and all the periphery agents are of type *b* (when $Q > 0$) or the disconnected core-periphery network where all the core agents are of type *a* and all the periphery agents are of type *b* (when $Q < 0$). We will first show that the efficient network has no pair of type *a* agents which are not directly connected. Let g be a network in which there is a disconnected pair of agents *i* and *j*, both of them are type *a* agents. Consider the network $g' = g + ij$. The minimal difference in total utility between g' and *g* is the differences in these two agents' utilities assuming that they have a path of length two between them and that this link does not improve any other shortest path in the network. Thus, the minimal difference for both agents is $2(\delta w_1 - c - \delta^2 w_1)$ which is positive since $(\delta - \delta^2)w_1 > c$. Therefore, for any network *g* in which there exists a disconnected pair of agents *i* and *j*, both of them are type *a* agents, there is a network $g' = g + ij$ with higher total utility. Thus, the efficient network belongs to the set of networks in which type *a* agents are completely connected. Next we will show that the one-gate minimally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* has the highest total utility among the set of connected networks. Consider the maximal overall value of a connected network in which type *a* agents are completely connected among themselves. In any such network there are $\frac{k(k-1)}{2}$ paths between two type *a* agents, kl paths between type *a* and type *b* agents and $\frac{l(l-1)}{2}$ paths between two type *b* agents. Since there is a complete sub-graph of the type *a* agents all their internal $\frac{k(k-1)}{2}$ paths are direct links. Denote by $K_1 \geq 1$ the number of direct links between type *a* and type *b* agents (it must be at least one since it is a connected network) and by $K_2 \geq 0$ the number of direct links between two type *b* agents. Thus, there are $kl - K_1$ indirect links between type *a* and type *b* agents and $\frac{l(l-1)}{2} - K_2$ indirect links between two type *b* agents. Note that since there are *l* type *b* agents it must be that $K_1 + K_2 \geq l$. The maximal overall value of this network is achieved when all the indirect links are of length two:

$$\frac{k(k-1)}{2}(2\delta w_1 - 2c) + K_1(2\delta w_2 - 2c) + K_2(2\delta w_3 - 2c) + 2\delta^2 w_2(kl - K_1) + 2\delta^2 w_3\left(\frac{l(l-1)}{2} - K_2\right)$$

The overall value of a one-gate minimally connected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* is $\frac{k(k-1)}{2}(2\delta w_1 - 2c) + l(2\delta w_2 - 2c) + 2\delta^2 w_2(k-1)l + 2\delta^2 w_3 \frac{l(l-1)}{2}$. The difference between the maximal value and the one-gate network is $2(K_1 - l)(\delta w_2 - \delta^2 w_2 - c) + 2K_2(\delta w_3 - \delta^2 w_3 - c)$. Note that this difference has to be non-negative since the maximal value has to be at least as high as the total utility of the one-gate minimally connected network. Therefore, it must be that $(l - K_1)(c - \delta w_2 + \delta^2 w_2) \geq K_2(c - \delta w_3 + \delta^2 w_3)$. Since $c > \delta w_2 > (\delta - \delta^2)w_2 > (\delta - \delta^2)w_3$, it holds that $c - \delta w_2 + \delta^2 w_2 < c - \delta w_3 + \delta^2 w_3$ and therefore it must be that either $l - K_1 > K_2$ or $K_1 = l$ and $K_2 = 0$. Note that the first option violates the connectivity condition $K_1 + K_2 \geq l$. In conclusion, the highest total utility among connected networks is achieved if this network has complete clique of all type *a* agents, no connections between type *b* agents and *l* connections between type *a* agents and type *b* agents. Due to connectivity it must be that the *l* links between type *b* agents and type *a* agents are divided such that each type *b* agent has exactly one such link. Therefore, the network belongs to the set of minimally connected core-periphery networks in which all the core agents are of type *a* and all the periphery agents are of type *b*. Moreover, we showed that the one-gate minimally connected core-periphery network achieves the maximum. To show that other minimally connected core-periphery networks do not achieve the maximum note that if all type *b* agents connect to the same type *a* agent it increases the utility from the indirect links within the type *b* agents and does not change the utility from other types of connections (the connections between type *a* agents are still of length one and *l* of the intertype connections are of length one and the rest are of length two). Hence, the network that achieves the highest total utility among all connected networks is the one-gate minimally connected core-periphery networks in which all the core agents are of type *a* and all the periphery agents are of type *b*. So far we showed that the efficient network is either the one-gate minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b* or a disconnected network in which all type *a* agents are completely connected among themselves and this *a*-component is a one-gate minimally connected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* (as a conclusion from the proof above). For the last step, let an *m*-one-gate network be a network in which all the type *a* agents are completely connected among themselves, *m* of the type *b* agents are connected to the same type *a* agent (the gate) and the rest of the type *b* agents are completely isolated. We will show that the $(l-1)$-one-gate network has higher total utility than all the disconnected networks in which there is at least one link outside the *a*-component. Let us explore the disconnected networks in which there is at least one link outside the *a*-component. Since the agents who are not connected to the *a*-component are all of type *b* we can use proposition 1 of Jackson and Wolinsky (1996) to assert that since $c > (\delta - \delta^2)w_3$ this group of agents will achieve its maximal total utility either as a star encompassing all the group members (*b*-star) or as an empty network. Thus, we have to show that the $(l-1)$-one-gate network has higher total utility than the double-component network that combines the *a*-component and the *b*-star. Let the number of agents in the *b*-star be *h* and let the utility of the *a*-component be X. The total utility of the network if the group is organized as a star is $X+2(h-1)(\delta w_3-c)+(h-1)(h-2)\delta^2 w_3$. The total utility of the network if all the leaves of the *b*-star replace their links from the *b*-star center to the gate of the *a*-component (to create an $(l-1)$-one-gate network) is at least $X+2(h-1)(\delta w_2-c)+(h-1)(h-2)\delta^2 w_3$ since the previous *b*-star center is now isolated (note that it is minimal since we do not count the indirect connections between the original members of the *a*-component and the $h-1$ newcomers). Since the second expression is larger, the total utility of the $(l-1)$-one-gate network is higher than all the disconnected networks in which there is at least one link outside the *a*-component. Note that so far we have shown that the *l*-one-gate network has higher utility than all the connected networks and that the $(l-1)$-one-gate has higher total utility than all the disconnected networks in which there is at least one link outside the *a*-component. Remember that as a conclusion from the previous proof above, the *m*-one-gate network has the highest utility among the set of networks in which the *a*-component includes *m* type *b* agents and there are no links among the other *l-m* type *b* agents. Thus, we have shown that the efficient network is the network that achieves the highest total utility among the set of *m*-one-gate networks ($m=0,\ldots,l$). The total utility of an *m*-one-gate network is $k(k-1)\delta w_1+2m\delta w_2+2(k-1)m\delta^2 w_2+m(m-1)\delta^2 w_3-k(k-1)c-2mc$. It is easy to see that the total utility of the *m*-one-gate network is an upward parabola in *m* and therefore its maximum is achieved on one of the edges – either $m=l$ (one-gate minimally connected core-periphery networks in which all the core agents are of type *a* and all the periphery agents are of type *b*) or $m=0$ (disconnected core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b*). By the definition of Q it is clear that the difference between the total utility of these networks equals exactly $2lQ$. Therefore, if $Q>0$ the set of one-gate minimally connected core-periphery networks in which all core agents are of type *a* and all periphery agents are of type *b* are strongly efficient and there are no other efficient networks (proposition 4.2.3) while if $Q<0$ the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is the unique strongly efficient network (efficiency part of proposition 4.1). It is left to show that when $Q<0$ there are no pairwise stable networks besides the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* (for this part of the proof we will denote this network by *d*) and to characterize the non core-periphery networks which are pairwise stable when $Q>0$. First, we will show that if $Q<0$ the unique pairwise stable network is *d*. Let g be another pairwise stable network. Therefore, either *g* has two type *a* agents which are not directly connected or it has a directly connected pair of agents, at least one of them is a type *b* agent. Let g' be a network in which there is a pair of type *a* agents who are not directly connected. In order to examine the minimal contribution of a direct link to these agents' utilities, assume that a path of length two links them and that connecting them directly does not shorten any of their other connections. Their net utility gains from a direct connection are $\delta w_1-c-\delta^2 w_1$. Since $(\delta-\delta^2)w_1>c$ it will surely be beneficial for these agents to connect directly, let alone if a longer path links them and/or if this link shortens their connections to other agents. Thus, g' is not pairwise stable. Let g'' be a pairwise stable network in which each pair of type *a* agents are directly connected and some of the type *a* agents are directly connected to type *b* agents. Let us compare the utilities of the agents in network *d* to their utilities in g''. Type *a* agents that are not connected directly to type *b* agents in g'' surely have higher utility in g'' than in *d* since they benefit from the indirect connections to type *b* agents without changing their costs. Type *a* agents that have direct connections to type *b* agents in g'' have the utility they had in *d* plus the utility they gain from their direct connections to type *b* agents. Using lemma 1, if g'' is pairwise stable, it is straightforward that the total relative contribution of each type *b* agent to his type *a* neighbor must be non-negative. Therefore, the total utility of each of the type *a* agents is at least as high in g'' as it is in *d*. Thus, the sum of type *a* agents' utilities in g'' is at least as high as it is in *d*. (note that it might be equal if there is only one type *a* agent in the network). Since in network *d* the total utility of type *b* agents is zero and since network *d* is uniquely efficient then there must be at least one type *b* agent in g'' with negative utility which contradicts, by the implication of lemma 1, the stability of g''. Thus, in a pairwise stable network when $Q < 0$ type agents are completely connected between themselves and completely disconnected from type *b* agents. Let g''' be a pairwise stable network in which each pair of type *a* agents are directly connected, there are no direct links between type *a* and type *b* agents and there is at least one pair of type *b* agents that are directly connected. Note that the sum of utilities of type *a* agents in g''' is equal to the sum of utilities of type *a* agents in network *d*. Thus, it must be that the sum of utilities of type b agents in g''' is negative, since *d* is uniquely efficient and the sum of utilities of type *b* agents in *d* is zero. Therefore, there is at least one type *b* agent in g''' that have negative utility which contradicts, by the implication of lemma 1, the stability of g'''. This completes the proof that if $Q < 0$ the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is the unique pairwise stable network.

Next, we will show some characteristics of pairwise stable networks when $Q > 0$. We conjecture, but fail to prove, that when $Q > 0$ and assumption 1 and 2 hold, the disconnected core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is the unique pairwise stable network. The following two examples show that when the heterogeneity condition is not satisfied the disconnected core-periphery network is not unique.

Consider the following network where the black circles stand for type *a* agents and white circles stand for type *b* agents. One can verify that under the values $c = \frac{161}{320}, \delta = \frac{1}{2}, w_1 = 3, w_2 = 1, w_3 = \frac{9}{10}$, the range and assumption 1 is satisfied while assumption 2 is violated. However, in this case this non-core-periphery network is pairwise stable. Moreover, is assumption 2 is violated there are pairwise stable networks which are neither connected nor disconnected core-periphery networks. One can verify, for example, that the network with two connected type *a* agents and a separate circle of eleven type *b* agents is pairwise stable in the given range under the following values: $c = \frac{25}{32}, \delta = \frac{1}{2}, w_1 = 4, w_2 = 1, w_3 = \frac{1600}{1921}$.



Next we will show that when $Q > 0$ any pairwise stable network, which is not the disconnected core-periphery network, is a non-core-periphery network in which all type *a* agents are directly connected to each other and there is no type *b* agent with exactly one direct connection. Let g' be a network in which there is a pair of type *a* agents who are not directly connected. In order to examine the minimal contribution of a direct link to these agents' utilities, assume that a path of length two links them and that connecting them directly does not shorten any of their other connections. Their net utility gains from a direct connection are $\delta w_1 - c - \delta^2 w_1$. Since $(\delta - \delta^2)w_1 > c$ it will surely be beneficial for these agents to connect directly, let alone if a longer path links them and/or if this link shortens their connections to other agents. Thus, g' is not pairwise stable, and the set of pairwise stable networks is a subset of the set of all networks in which each pair of type *a* agents are directly connected. Let g'' be a network in which there is a type *b* agent, agent *i*, who has exactly one direct connection. Consider, agent *j* who is the only agent with a direct connection to agent *i*. agent *j*'s net utility from the direct link to agent *i* is $\delta w_k - c(k \in \{2,3\})$. Since $\max_{k \in \{2,3\}} \delta w_k - c = \delta w_2 - c < 0$, agent *j*, whatever her type is, would prefer to severe the link and therefore g'' is not pairwise stable. Next we will show that the disconnected core-periphery network is the only pairwise stable core-periphery network. Let g''' be a non-disconnected core-periphery network is which all the core agents are of type *a* and all the periphery agents are of type *b*. Hence, all the type *a* agents in g''' are directly connected between themselves, while all the type *b* agents are directly disconnected among themselves and there is at least one type *b* agent who is not isolated and is connected directly to type *a* agent. From the proof above, we can deduce that any type b agent who is not isolated has at least two direct connections to type *a* agents. However, since the type *a* agents are completely connected they can severe the link to this type *b* agent and still have a path of length two connecting them to her. They will prefer to do so since $(\delta - \delta^2)w_2 < c$. Therefore, g''' is not pairwise stable. Thus, we showed that any pairwise stable network is either the disconnected core-periphery network or a non-core-periphery network in which all type *a* agents are directly connected to each other and there is no type *b* agent with exactly one direct connection.

Proof of proposition 5

First we will show that no core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is pairwise stable under the given range if assumption 1 holds. Let *g* be a core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b* and there is at least one direct link between type *a* and type *b* agents. Assume that g is pairwise stable. By the definition of core periphery networks there are no direct links between type *b* agents and every pair of type a agents maintains a direct link. Let agent *i* be a type *a* agent and agent *j* be a type *b* agent, such that $ij \in g$. Thus, the benefit that agent *i* receives from this link is $\delta w_2 - c$ since this link provides agent *i* with no shorter paths except the one to agent *j*. Since $c > (\delta - \delta^2)w_1$, assumption 1 guarantees that $c > \delta w_2$ and therefore the benefit of this link to agent *i* is negative and he would like to severe it. Therefore g is not pairwise stable. It is left to show that the disconnected core periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is not pairwise stable. Assume that it is pairwise stable. The net benefit of agent *i* of type *a* from the direct link to another agent *j*, also of type *a*, is $\delta w_1 - c$ since this link does not shorten any of his other paths. If agent *i* drops the link to agent *j* he has a path of length two to agent *j* through a third type *a* agent ($k \geq 3$) which yields $\delta^2 w_1$. Since $c > (\delta - \delta^2)w_1$, agent *i* would like to severe his link to agent *j*. This contradicts the assumption that the disconnected core periphery network in which all core agents are of type *a* and all periphery agents are of type *b* is pairwise stable. We showed that if $c > (\delta - \delta^2)w_1$, $k \geq 3$ and assumption 1 holds, then there is no pairwise stable core periphery network in which all core agents are of type *a* and all periphery agents are of type *b*. Note that if assumption 1 is violated then this statement is false. One can verify that under the values $c = \frac{1}{2}, \delta = \frac{3}{4}, w_1 = 2, w_2 = 1, w_3 = \frac{1}{4}, k = 3, l = 6$, the linking costs range and the minimal number of type *a* agents are satisfied, assumption 1 is violated and the minimally connected core periphery network in which all core agents are of type *a*, all periphery agents are of type *b* and each type *a* agent is directly connected to two type *b* agents is pairwise stable. Now we will show that the efficient network is never a core-periphery network in which all core agents are of type *a* and all periphery agents are of type *b*. Denote by A^g the set of type *a* agents in network *g*. Denote by B_c^g the set of type *b* agents who are not isolated in network *g* and by B_i^g the set of type *b* agents who are isolated in network *g*. Let $g \in CP_{p,q}$ if *g* is a core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b*, it has *p* links between type *a* agents and type *b* agents and $q = \left|B_c^g\right|$[37] (Naturally, $p \geq q$). Let $g \in HCP_{p,q}$ if $g \in CP_{p,q}$ and every pair of agents in B_c^g have a length two path between them (they have at least one common core neighbor). Let $g \in GCP_{p,q}$ if $g \in CP_{p,q}$ and there is a member of A^g (the "gate") which is directly connected to all the agents in B_c^g. By definition, $\phi \neq GCP_{p,q} \subset HCP_{p,q} \subset CP_{p,q}$. In what follows we will divide the total utility of a given network into four components: the utility from the connections between two type *a* agents, the utility from connections between two type *b* agents, the utility from connections between type *a* and type *b* agents and the total linking costs[38]. Given *p* and *q*, the difference in total utility among the members of $CP_{p,q}$ comes solely from the connections between pairs of type *b* agents. This is true since the type *a* agents are completely connected, the total linking costs are identical between all

[37] The disconnected core periphery network belongs to $CP_{0,0}$, the minimally connected core periphery networks belong to $CP_{l,l}$ and the maximally connected core periphery network belongs to $CP_{lk,l}$.

[38] $$v(g) = \sum_{i \in N} u_i(g) = \sum_{i \in N}\left[\sum_{j \in N \setminus \{i\}} \delta^{d_{ij}} f(t_i, t_j) - \sum_{j \in N \setminus \{i\}: ij \in g} c\right] = \sum_{\substack{(i,j) \in N \times N \\ i \neq j}} \delta^{d_{ij}} f(t_i, t_j) - \sum_{(i,j) \in N \times N : ij \in g} c$$

$$v(g) = \sum_{\substack{(i,j) \in N \times N \\ i \neq j \\ t_i = a, t_j = a}} \delta^{d_{ij}} w_1 + \sum_{\substack{(i,j) \in N \times N \\ i \neq j \\ t_i = a, t_j = b}} \delta^{d_{ij}} w_2 + \sum_{\substack{(i,j) \in N \times N \\ i \neq j \\ t_i = b, t_j = b}} \delta^{d_{ij}} w_3 - \sum_{(i,j) \in N \times N : ij \in g} c$$

these networks since the number of connections is fixed, all these networks possess the same number of direct links between type *a* agents and type *b* agents and therefore also the same number of length two connections (there are no longer shortest paths between type *a* and type *b* agents due to the complete connectivity of the core). Moreover, since the members of B_i^g do not contribute anything to the total utility, the difference in total utility between the members of $CP_{p,q}$ are solely due to the internal connections of the members of B_c^g. Since the networks in $CP_{p,q}$ are core periphery networks such that all core agents are of type *a* and all periphery agents are of type *b*, they possess no direct links between type *b* agents. Therefore the highest total utility for members of $CP_{p,q}$ is achieved by the members of $HCP_{p,q}$ since every pair of connected type *b* agent has a length two path. In particular, this value is achieved by all the members of $GCP_{p,q}$. Now we will show that the value achieved by the members of $GCP_{q,q}$ is higher than the value achieved by the members of $GCP_{p,q}$ for all $p > q$. Given *q*, the differences in total utility between the members of those groups comes solely from the benefits and costs of connections between type *b* agents and type *a* agents different from the gate. This is true since the type *a* agents are completely connected, the type b agents have length two path among themselves and the total linking costs excluding the connections between type *b* agents and type *a* agents different from the gate are identical. The members of $GCP_{q,q}$ have no links between type *b* agents and type *a* agents different from the gate and therefore their additional net benefit is zero. However, the members of $GCP_{p,q}$ for $p > q$ have at least one link between type *b* agent and type *a* agent different from the gate. Any such link does not change the utility of other agents since they already have a path of length of at most two with both types of agents. Therefore, the contribution of this link to the total utility comes only from the shortening of the path between those agents from a length two to a length one and each of them pays *c* for that benefit. Therefore its total contribution is $2\left[\left(\delta-\delta^2\right)w_2-c\right]$ which is negative. Thus, given *q*, we showed that for every $p > q$ the total utility of the members of $GCP_{p,q}$ is lower than the total utility of the members of $GCP_{q,q}$ by $2(p-q)\left[\left(\delta-\delta^2\right)w_2-c\right]$. Since for all $p > q$ no member of $CP_{p,q}$ achieves higher total utility then the members of $GCP_{p,q}$, we showed that none of these networks is efficient. It is left to show that for every *q* the members of $CP_{q,q}$ are not efficient. Actually, since the members of $HCP_{q,q}$ achieve the highest total utility among the members of $CP_{q,q}$, it is left to show that for every *q* the members of $HCP_{q,q}$ are not efficient. The general architecture of these networks is of a core including all type *a* agents completely connected among themselves, *q* type *b* agents with single link to a type *a* agent (the "gate") and *l-q* isolated type *b* agents. We will show that for every *q* there is a non core periphery architecture which yields higher total utility, namely networks in which the type *a* agents form a star around the gate while the type *b* agents do not change their linking scheme (all-star architecture). Note that the utilities of the type *b* agents in both architectures are the same – the isolates have zero utility, the non isolates have one direct link to a type *a* agent and length two paths to all other non isolated agents (of both types). The utilities of the type *a* agents from the connections with type *b* agents are also identical – the gate is connected directly to each of the type *b* agents while the others have paths of length two to each of them. Another unchanged component is the utilities from the links of the non gate type *a* agents with the gate. Thus, the difference in utility comes from the benefits and costs of the internal connections of the type *a* agents who are not the gate. In the architecture of the $HCP_{q,q}$'s members the contribution to total utility from a connection of two non gate type a agents is $2[\delta w_1 - c]$ while the contribution to total utility from a connection of two non gate type a agents in the all-star architecture is $2\delta^2 w_1$ ($k \geq 3$). Since $c > \left(\delta-\delta^2\right)w_1$ the total contribution of such a connection is higher in the all-star architecture. Therefore, for every *q* the non core periphery all-star architecture yields higher total utility than the members of $HCP_{q,q}$. Hence, we found that every core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* has a network which has higher total utility. Thus, we showed that core-periphery network in which all the core agents are of type *a* and all the periphery agents are of type *b* are not efficient.

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