Time-to-build and the Inverse U-Shape Investment-Uncertainty Relationship

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Abstract

The effect that investment lags have on the uncertainty-investment relationship is studied by modifying the Bar-Ilan and Strange (1996) model in a manner that enables analytical solution. It turns out that: (i) If the time lag is sufficiently small, uncertainty affects investment negatively; (ii) A sufficiently large time lag engenders an inverse U-shape relationship between the degree of uncertainty and the profit level that triggers investment; (iii) When such an inverse U-shape exists, the higher is the length of the time lag (or the degree of profit convexity) the wider is the range of a positive uncertainty-investment relationship.

Keywords: Investment, Uncertainty, Time to build

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Introduction

Under the classical conditions of competition, the profit of the firm is a convex function of the output price.[1] Relying on this convexity, Hartman (1972) has shown that output price uncertainty exerts a positive effect on investment. Specifically, he has shown that a mean preserving spread in output prices increases the expected profitability of the investment. The rationale underlying this increased profitability is that due to this convexity the increase in profits corresponding to a possible certain increase in the price is larger than the decrease in profits that would follow a decline of the same size in the output price. Later, Bernanke (1983), McDonald and Siegel (1986), Dixit (1989) and others have shown that despite this convexity uncertainty affects investment negatively if the firm can choose the investment timing optimally.[2] In the most recent swing of this pendulum, Bar-Ilan and Strange (1996) have shown that introducing time-to-build (in the form of a time lag between the moment of investment and the moment in which profits starts to accrue) into Dixit's (1989) model enables a positive effect of uncertainty on investment.

The explanation to the result of Bar-Ilan and Strange (1996) is that the introduction of this time lag pulls the rug from under Bernanke's Bad News Principle that underlies the negative effect uncertainty has on investment. According to this principle, "Good News" regarding the investment are irrelevant to the investment timing. Thus, increased uncertainty, which makes the "Bad News" worse and the "Good News" better, affects investment timing only via the worsening of the "Bad News". The reason why "Good News" is irrelevant to the timing of investment is that

[1] The reason for thus convexity is that the higher the price the larger the quantity that the firm produces. Therefore, for a given increase in the output price, the higher the initial price the larger the additional revenue - and profits - from selling the quantity that corresponds to this higher price (An increase in price also entails the production of more output but if the increase in price is sufficiently small then this has a negligible effect on the profit since the marginal profit of the optimizing firm is zero).
[2] For detailed surveys of this literature see Pindyck (1991) or Dixit and Pindyck (1994).

in the absence of time-to-build the proceeds attached to them can be collected the minute they are realized and therefore are collected both in the case of early investment and in the case when investment is postponed until the arrival of these "Good News". Introducing time-to-build makes it impossible for the firm to receive the proceeds of "Good News" right from the minute they are realized and therefore restores their relevancy to the investment timing decision.

Note that introducing an investment lag to the model is not sufficient for uncertainty to have a positive effect on investment unless the profit is a convex function of the output price. In Bar-Ilan and Strange (1996), as in Dixit (1989), this convexity is borne out by the option the firm has to abandon the investment, an option that the firm exercises if the output price is sufficiently low.

Since Bar-Ilan and Strange's (1996) model can only be solved numerically, their analysis is limited to showing that for some parameter values uncertainty affects investment positively. The purpose of the current article is to broaden our knowledge of how time-to-build affects the uncertainty-investment relationship beyond that. To this end I use here a version of the Bar-Ilan and Strange (1996) model modified as follows. First, the exit option is deleted in order to enable an analytical solution. Second, in order to restore the convexity of the profit function in the output price, the Bar-Ilan and Strange (1996) assumption that the production process generates a flow of fixed quantity is replaced by the weaker assumption that the firm can vary its output according to market conditions. These modifications lead to the analytical derivation of the following results:

- For a sufficiently small time lag, uncertainty affects investment negatively.
- A sufficiently large time lag leads to an inverse U-shape relation between the degree of uncertainty and the profit level that triggers investment.
- When such an inverse U-shape relation exists, the longer the time lag (or the larger the degree of profit convexity) the wider the range of positive uncertainty-investment relationship.

Although the focus of the current article is on the qualitative connection between uncertainty and investment, it is important to emphasize that at least since the seminal work of Kydland and Prescott (1982) time-to-build is widely recognized as a quantitatively important feature of the business cycle, with a particular role in shaping investment decisions. In particular, construction lags cannot be left out of the study of the effect that uncertainty has on irreversible investment because, naturally, an investment that costs a large irreversible sum should also take an accordingly long time to build. Following this rationale, Bar-Ilan and Strange (1996, pages 610 and 619-620) bring several references to the quantitative importance of time-to-build in industries such as the pharmaceutical, power generation, aerospace, bulk chemicals or office-buildings industries.

The convexity of the profit function in the output price is another building block of this paper that enjoys a wide empirical support for its quantitative importance. There are several sources for this convexity, like the existence of an exit option or the use of production margins such as shiftwork or extra hours.[3] A third source for this profit convexity is the ability of the firm to vary the quantity it

[3] Using shiftwork or extra-hours implies that production is increased without a fall in the marginal productivity of labor. Reducing thus the concavity in production strengthens the convexity of the profit function in the output price. See Mayshar and Halevy (1997) for a theoretical analysis and a survey of the literature on the extensive use done by these production margins.

produces in response to changes in market prices. For simplicity this is the sole source for the profit convexity in the theoretical model analyzed in this article. Put in terms of the elasticity of production with respect to labor - while in most articles of the relevant literature the value of this elasticity is assumed to be zero, here it is assumed to be strictly positive. This positive elasticity makes the profit a convex function of the output price, where the larger the elasticity the larger the degree of profit convexity.

A large amount of empirical work shows that the production-labor elasticity is rather large – sometimes even close to unity – implying a large amount of profit convexity.[4] Tables 3.2 and 3.3 in Hammermesh (1993), for example, presents industry-level estimates of this elasticity based on a summery of a large number of industry studies. The estimates are often close to unity, implying a closeness to linear production and therefore to infinitely large degree of profit convexity in output price.

The article is organized as follows. Section 2 presents the model and its analysis. Section 3 offers some concluding remarks. Some of the technical proofs where relegated to an appendix.

2. The Model

Time in the model is continuous. Consider an infinitely lived, risk-neutral firm that can enter a project in which it produces output according to:

(1) $$Q_t = AL_t^{\alpha}$$

[4] A production-labor elasticity not below 1 contradicts with decreasing marginal labor productivity.

Where Q_t and L_t are, respectively, the instantaneous output and labor input of the production process and A and α are constants satisfying $A > 0$ and $0 \leq \alpha < 1$.[5] There is no cost for adjusting the amount of labor the firm employs.[6]

By standard optimization, if the firm enters the project its instantaneous profit (π_t), given the output price (P_t) and the labor wage (x), satisfies:

$$\text{(2)} \qquad \pi_t = CP_t^{\gamma}$$

where C and γ are constants defined by:

$$\gamma \equiv \frac{1}{1-\alpha} \qquad\qquad C \equiv \left(\frac{A}{x^{\alpha}}\right)^{\gamma}\left(\alpha^{\alpha\gamma} - \alpha^{\gamma}\right).$$

Note that $\gamma \geq 1$ since $0 \leq \alpha < 1$. Also note that $\gamma'(\alpha) > 0$, implying that the higher α the higher the convexity of π in P. Finally note that $C \geq 0$ since $\gamma \geq 1$ and $0 \leq \alpha < 1$.

To enter the project the firm must incur the cost $k > 0$. A lag of length $h \geq 0$ exists between the time in which the firm pays the entry cost and the time in which the project becomes active, where the term "active" means that profits start to accrue. The discount rate of the firm is denoted by ρ. After the firm enters the project it cannot exit it. ρ, k and x are constants. The uncertainty arises from the output price, P_t, which evolves exogenously over time according to the rule:

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[5] The case analyzed by Bar-Ilan and Strange (1996) and Dixit (1989) is that of fixed quantity, which corresponds to $\alpha = 0$.

[6] This is merely a simplifying assumption, as a vast literature has already established the importance of labor adjustment costs to the analysis of the firm level behavior. See Hamermesh and Pfann (1996) for an analysis and a survey.

(3) $$dP_t = \mu P_t dt + \sigma P_t dz,$$

where $\sigma > 0$ and dz is the increment of a standard Wiener process, uncorrelated across time and at any one instant satisfying $E(dz) = 0$ and $E(dz^2) = dt$. This means that P_t is a geometric Brownian Motion. By Itô's lemma and (2), π_t is a geometric Brownian motion too, with the constant parameters:

(4) $$\mu_\pi \equiv \gamma [\mu + ½ (\gamma - 1)\sigma^2] \qquad \sigma_\pi \equiv \gamma\sigma$$

Convergence of the firm's expected net present value requires the assumption $\mu_\pi < \rho$, which means that σ must satisfy:

(5) $$\sigma < \bar{\sigma} \equiv \sqrt{2\frac{\rho - \gamma\mu}{\gamma(\gamma - 1)}}$$

Thus constructed, the model closely resembles the model solved by Bar-Ilan and Strange (1996). The three differences between these models are: (i) Their model contains an option to exit the project by paying a fixed exit cost denoted by l. The no exit case analyzed here corresponds to their analysis of the specific case where l approaches infinity; (ii) Their model contains a flow of a production cost with constant magnitude that they denote as w. The model analyzed here corresponds with the specific case in their model where $w = 0$; (iii) In their model the instantaneous output is assumed constant at unity and therefore the instantaneous profit is $P_t – w$. Assuming $w = 0$ renders the profit flow in both models (P_t in theirs and π_t here) a geometric Brownian Motion. Since the only property of P_t relevant to their solution procedure is its being a geometric Brownian Motion, it is possible to use their analysis in pages 612 – 615 by replacing P, μ and σ by π, μ_π and σ_π, respectively, and assuming that $w = 0$ and that l approaches infinity. The results are that the optimal policy is to enter once the profit process, π_t, reaches a certain threshold level denoted by π_H^h and given by:[7]

(6) $$\pi_H^h = \frac{\beta}{\beta - 1}(\rho - \mu_\pi)e^{-\mu_\pi h}k$$

where β is the positive root of the quadratic:

(7) $$½\sigma_\pi^2\beta^2 + (\mu_\pi - ½\sigma_\pi^2)\beta - \rho = 0$$

Applying the values of 0 and 1 for β in this quadratic reveals that one of its roots is negative and the other, β, exceeds unity. For brevity of notations, β' and β'' denote the first and second derivatives of β with respect to σ^2.

2.1 With no time lag

In this sub-section no time lag between paying the entry cost and the start of production exists, i.e., $h = 0$. By (6), the entry threshold in that case is:

(8) $$\pi_H = \frac{\beta}{\beta - 1}(\rho - \mu_\pi)k$$

[7] Equation (6) here is in fact equation (12) in Bar-Ilan and Strange (1996).

Differentiating with respect to σ^2 yields:

(9) $$\frac{\partial \pi_H}{\partial \sigma^2} = \left[\frac{-1}{(\beta-1)^2} \beta'(\rho - \mu_\pi) - \frac{\beta}{\beta-1} \frac{\partial \mu_\pi}{\partial \sigma^2} \right] k$$

$$= \frac{\beta}{\beta-1}(\rho - \mu_\pi)\left[\frac{-\beta'}{(\beta-1)\beta} - \frac{\gamma(\gamma-1)}{2(\rho-\mu_\pi)} \right] k \;=\; \pi_H \cdot f(\mu,\, \sigma^2,\; \rho,\, \gamma)$$

where:

(10) $$f(\mu, \sigma^2, \rho, \gamma) \equiv \frac{-\beta'}{(\beta-1)\beta} - \frac{\gamma(\gamma-1)}{2(\rho-\mu_\pi)}$$

The following lemma presents some properties of $f(\mu,\, \sigma^2,\, \rho,\, \gamma)$.

<u>*Lemma 1:*</u> $f(\mu,\, \sigma^2,\, \rho,\, \gamma)$ satisfies:

(a) $$\frac{\partial f(\mu, \sigma^2, \rho, \gamma)}{\partial \sigma^2} < 0 \qquad \forall\ \sigma \in (0,\ \bar{\sigma})$$

(b) $$\underset{\sigma \to 0}{Lim}\ f(\mu, \sigma^2, \rho, \gamma) = \begin{cases} \dfrac{\gamma}{2\mu} & If \quad \mu > 0 \\ -\dfrac{\gamma(\rho-\mu)}{2\mu(\rho - \gamma\mu)} & if \quad \mu \le 0 \end{cases} \equiv f^{*}(\mu, \rho, \gamma) > 0$$

(c) $$\underset{\sigma \to \bar{\sigma}}{Lim}\ f(\mu, \sigma^2, \rho, \gamma) = \frac{\gamma^3(\gamma-1)(\rho-\mu)}{2(2\gamma\rho - \gamma^2\mu - \rho)^2} \equiv f^{**}(\mu, \rho, \gamma) \ge 0$$

(d) $$\frac{\partial f(\mu, \sigma^2, \rho, \gamma)}{\partial \gamma} < 0$$

(e) $f(\mu,\, \sigma^2,\, \rho,\, \gamma) > 0 \qquad \forall\ \sigma \in (0,\ \bar{\sigma})$

Proof: in the appendix. □

Figure 1 below depicts $f(\mu, \sigma^2, \rho, \gamma)$ based on *Lemma 1*.

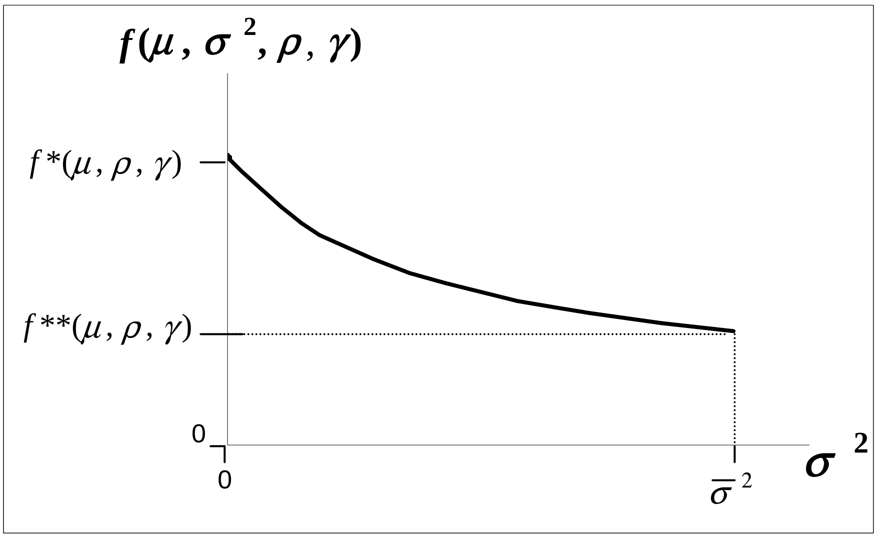

Figure 1: $f(\mu, \sigma^2, \rho, \gamma)$ as a function of σ^2.

The immediate corollary from *lemma 1* and (9) is that $\frac{\partial \pi_H}{\partial \sigma^2} > 0$ throughout the relevant range of σ. Thus, in the absence of a time lag uncertainty affects entry negatively, despite the "a-la Hartman" price convexity of the profit function.

2.2 With a time lag

Returning to the case of a time lag and it follows from (8) and (6) that:

(11) $$\pi_H^h = e^{-\mu_\pi h} \pi_H$$

where π_H is the value of π_H^h when $h = 0$, as given by (8). Differentiating π_H^h with respect to σ^2 yields:

(12) $$\frac{\partial \pi_H^h}{\partial \sigma^2} = e^{-\mu_\pi h}\frac{\partial \pi_H}{\partial \sigma^2} - \frac{\partial e^{-\mu_\pi h}}{\partial \sigma^2}\pi_H$$

$$= \pi_H e^{-\mu_\pi h}\left[f\left(\mu, \sigma^2, \rho, \gamma\right) - h\frac{\gamma(\gamma-1)}{2}\right]$$

where the second equality follows from (9) and (4).

The following two propositions show that $\frac{\partial \pi_H^h}{\partial \sigma^2}$ might be negative, provided that π is convex in *P*, i.e., that $\gamma > 1$. They also show that the range of values of σ^2 in which $\frac{\partial \pi_H^h}{\partial \sigma^2} < 0$ is an increasing function of *h* and γ, i.e., that a positive relation between entry and uncertainty becomes more likely as the investment lag or the degree of convexity rise.

Proposition 1: If $\gamma = 1$ then $\frac{\partial \pi_H^h}{\partial \sigma^2} > 0$ throughout the relevant range of σ.

Proof: If $\gamma = 1$ then, by part (c) of *Lemma 1*, $\underset{\sigma \to \bar{\sigma}}{Lim}\, f\left(\mu, \sigma^2, \rho, \gamma\right) = 0$. Therefore, from part (a) of *Lemma 1* it follows that $f(\mu, \sigma, \rho, \gamma) > 0$ for all $\sigma \in (0, \bar{\sigma})$. Thus, by (12), $\frac{\partial \pi_H^h}{\partial \sigma^2} > 0$ for all $\sigma \in (0, \bar{\sigma})$. □

Note that γ has the value of unity when α is zero, implying by (1) that supply is inelastic at the quantity *A*. In that case the profit is not a convex function of the output price because increasing the output price by the amount $|\Delta P|$ or lowering the output price by the same $|\Delta P|$ would, respectively, increase or decrease revenues and profits by the same amount of $A\cdot|\Delta P|$. Thus, a mean preserving spread would make the "bad news" worse and the "good news" better by the same amount in this case. Still, increased uncertainty has a negative effect on the propensity of the firm to invest despite this symmetry. The reason for that is that only part of the "good news" benefits, the part that cannot be collected because of the time lag, is relevant to the dilemma the firm has between entering immediately or delaying entry until the "good news" would arrive. Convexity, therefore, is mandatory for $\frac{\partial \pi_H^h}{\partial \sigma^2}$ to be negative. Under convexity a mean preserving spread in price uncertainty makes the "good news" better by more than it makes the "bad news" worse, and therefore might affect investment positively if the time lags makes a sufficiently large part of the "good news" relevant to the investment timing dilemma of the firm. This requires a sufficiently long time lag, as established by the following *proposition 2*.

For brevity, the following proposition makes use of the definitions:

(13) $$h^{**}(\mu,\rho,\gamma) \equiv 2\frac{f^{**}(\mu,\rho,\gamma)}{\gamma(\gamma-1)} \qquad h^{*}(\mu,\rho,\gamma) \equiv 2\frac{f^{*}(\mu,\rho,\gamma)}{\gamma(\gamma-1)}.$$

<u>*Proposition 2*</u>: If $\gamma > 1$ then:

(a) If $h \le h^{**}(\mu, \rho, \gamma)$ then π_H^h is increasing in σ^2 throughout the range $\sigma \in (0, \bar{\sigma})$.

(b) If $h^{**}(\mu, \rho, \gamma) < h < h^{*}(\mu, \rho, \gamma)$ then π_H^h is an inverse U-shape function of σ within the range $(0, \bar{\sigma})$.

(c) If $h > h^{*}(\mu, \rho, \gamma)$ then π_H^h is decreasing in σ^2 throughout the range $\sigma \in (0, \bar{\sigma})$.

(d) $\sigma^{*}(\mu, \rho, \gamma, h)$ is decreasing in h and in γ.

Proof: follows directly from (12) and *lemma 1*. □

Part (d) of this proposition implies that $\sigma^*(\mu, \rho, \gamma, h) = 0$ when $h > h^*(\mu, \rho, \gamma)$. Figure 2 below shows σ^* as a function of h.

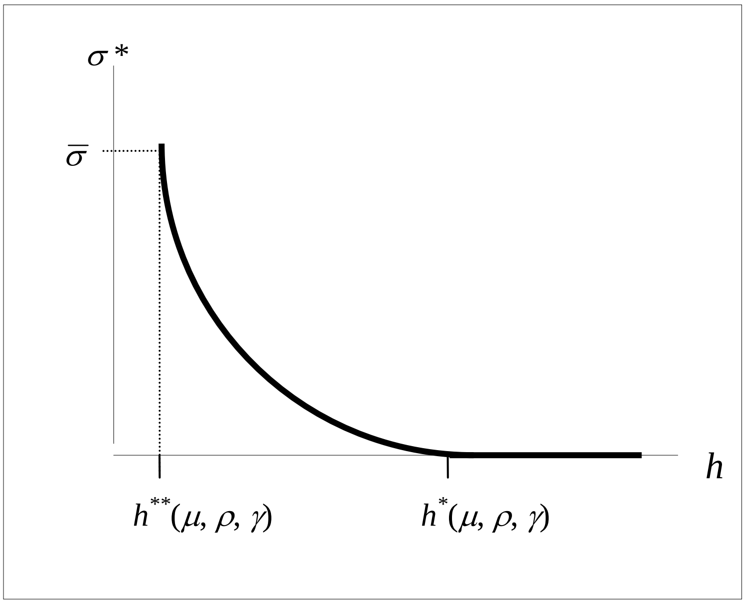

Figure 2: σ^* as a function of h. The higher the investment lag, the lower the level of σ^2 from which π_H decreases in σ^2. The larger the convexity of the profit function in output prices more to the left the location of this σ^* curve.

3. Concluding Remarks

In this article I have studied the effect of time-to-build on the uncertainty-investment relationship in a model when investment can be delayed. It was shown that if the time lag between the moment of investment and the moment when profits start to accrue is sufficiently small then uncertainty affects investment negatively, as the related literature usually shows. However, when this time lag is sufficiently long, an inverse U-shape relationship exits between uncertainty and investment.

A thorough analytical understanding of the effect of time-to-build on the uncertainty-investment relationship should be helpful to future empirical work. As this paper has shown, empirical models of investment under uncertainty should not analyze the effect of time-to-build in separation from other factors, but rather in interaction with the qualitative nature of the uncertainty-investment relationship.

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Appendix

A. Properties of β

Applying (4) in (7) yields that β is the positive root of:

(a.1) $$\tfrac{1}{2}\gamma^2\sigma^2\beta^2 + (\gamma\mu - \tfrac{1}{2}\gamma\sigma^2)\beta - \rho = 0$$

<u>*Lemma 2:*</u> β satisfies the following:

(a) $\beta' < 0 \quad \forall\ \sigma \in (0, \bar{\sigma})$

(b) $\beta'' > 0 \quad \forall\ \sigma \in (0, \bar{\sigma})$

(c) $\lim_{\sigma\to\bar{\sigma}} \beta = 1$

(d) $$\lim_{\sigma\to 0}\beta = \begin{cases} \dfrac{\rho}{\gamma\mu} & if \quad \mu > 0 \\ \infty & f \quad \mu \le 0 \end{cases}$$

(e) $$\lim_{\sigma\to\bar{\sigma}} \beta' = -\frac{\gamma(\gamma-1)^2}{2\left(2\gamma\rho - \rho - \gamma^2\mu\right)}$$

(f) $$\lim_{\sigma\to 0}\beta' = \begin{cases} -\dfrac{\rho(\rho-\mu)}{2\gamma\mu^3} & if \quad \mu > 0 \\ -\infty & f \quad \mu \le 0 \end{cases}$$

Proof: (c) and (d) follow directly from (a.1). By implicit derivation of (a.1):

(a.2) $$\beta' = -\frac{\frac{1}{2}\gamma^2\beta^2 - \frac{1}{2}\gamma\beta}{\gamma^2\sigma^2\beta + \gamma\mu - \frac{1}{2}\gamma\sigma^2} = -\tfrac{1}{2}\gamma\beta^2 \frac{\gamma\beta - 1}{2\rho - \gamma\left(\mu - \frac{1}{2}\sigma^2\right)\beta} < 0$$

where the second equality follows from (a.1). $\beta' < 0$ follows from the first equality for the case of $2\mu \geq \sigma^2$ and from the second equality for the case of $2\mu < \sigma^2$, taking into account in both cases that $\beta > 1$. This proves (a).

Taken together, (a) and (d) imply that if $\mu > 0$ then $\beta < \frac{\rho}{\gamma\mu}$. Based on (a.2):

(a.3) $$\beta'' = -\frac{(2\gamma\beta - 1)\beta'\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right) - \beta(\gamma\beta - 1)\left(\gamma\beta + \gamma\sigma^2\beta' - \frac{1}{2}\right)}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2}$$

$$= -\beta' \frac{(2\gamma\beta - 1)\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right) - \beta(\gamma\beta - 1)\gamma\sigma^2 + 2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)\left(\gamma\beta - \frac{1}{2}\right)}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2}$$

$$= -\beta' \frac{6\rho - 2\gamma\beta\mu - 2\mu + \sigma^2}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2}$$

where the second equality follows from (a.2) and the third equality follows from tedious, yet straightforward, algebra. If $\mu \leq 0$ then all terms in numerator are positive and therefore $\beta'' > 0$. If $\mu > 0$ then the numerator depends negatively on β and therefore, since $\beta < \frac{\rho}{\gamma\mu}$, in that case:

(a.4) $$\beta'' > -\beta' \frac{6r - 2\gamma \frac{\rho}{\gamma\mu}\mu - 2\mu + \sigma^2}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2} = -2\beta' \frac{4\rho - 2\mu + \sigma^2}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2} > 0$$

where the inequality follows from $\rho > \mu_\pi$ and from $\beta' < 0$. This proves (b). The proof of (e) follows directly from (a.2) and from (c). The proof of (f) follows from the second equality of (a.2) together with (d). □

B. Proof of *lemma 1*

Based on (4) and (10):

(a.5) $$\frac{\partial f\left(\mu, \sigma^2, \rho, \gamma\right)}{\partial \sigma^2} = -\frac{\beta''(\beta - 1)\beta - \beta'(2\beta - 1)}{(\beta - 1)^2 \beta^2} - \frac{\gamma^2(\gamma - 1)^2}{4(\rho - \mu_\pi)^2} < 0$$

where the inequality follows from parts (a) and (b) of *lemma 2* together with the results that $\beta > 1$ and $\gamma \geq 1$. Thus (a) is proved. The proof of (b) for the case of $\mu > 0$ stems from applying parts (f) and (d) of *lemma 2* in (10). For the case of $\mu \leq 0$ the proof of (b) is done by applying the second equality of (a.2) in (10). In order to prove (c) it is useful to present (10) as:

(a.6) $$f(\mu, \sigma^2, \rho, \gamma) \equiv \frac{\frac{-\beta'}{\beta} - \frac{\beta - 1}{\bar{\sigma}^2 - \sigma^2}}{\beta - 1}$$

As σ approaches $\bar{\sigma}$ both numerator and denominator of $\frac{\beta - 1}{\bar{\sigma}^2 - \sigma^2}$ approach 0 since β approaches 1. Using L'Hôpital's rule yields that in that case both the numerator and the denominator of $f(\mu, \sigma^2, \rho, \gamma)$ approach 0. Thus, by using L'Hôpital's rule again:

(a.7) $$\lim_{\sigma \to \bar{\sigma}} f\left(\mu, \sigma^2, \rho, \gamma\right) = \lim_{\sigma \to \bar{\sigma}} \frac{-\frac{\beta'' \beta - \beta'^2}{\beta^2} - \frac{\beta'\left(\bar{\sigma}^2 - \sigma^2\right) + (\beta - 1)}{\left(\bar{\sigma}^2 - \sigma^2\right)^2}}{\beta'}$$

Since both numerator and denominator of the second term in the main numerator approach zero, a repeated use of L'Hôpital's rule is needed:

(a.8) $$\lim_{\sigma \to \bar{\sigma}} f\left(\mu, \sigma^2, \rho, \gamma\right) = \lim_{\sigma \to \bar{\sigma}} \frac{-\frac{\beta'' \beta - \beta'^2}{\beta^2} - \frac{\beta''\left(\bar{\sigma}^2 - \sigma^2\right) - \beta' + \beta'}{-2\left(\bar{\sigma}^2 - \sigma^2\right)}}{\beta'}$$

$$= \lim_{\sigma \to \bar{\sigma}} \left(\beta' - \frac{\beta''}{2\beta'}\right) = \frac{\gamma^3(\gamma - 1)}{2} \frac{\rho - \mu}{\left(2\gamma\rho - \gamma^2\mu - \rho\right)^2} \geq 0$$

where the second equality is based on part (c) of *lemma 2*, the third equality springs from (a.3) together with part (e) of *lemma 2* and the inequality follows from the assumption that $\rho > 2\mu + \sigma^2$. This proves (c).

To prove (d) first note by implicit derivation of (a.1) that:

(a.9) $$\frac{\partial \beta}{\partial \gamma} = -\frac{\beta}{\gamma}$$

and therefore:

(a.10) $$\frac{\partial \beta'}{\partial \gamma} = \frac{\partial^2 \beta}{\partial \sigma^2 \partial \gamma} = -\frac{\beta'}{\gamma} > 0$$

Applying (a.10) in a differentiation of $f(\mu, \sigma^2, \rho, \gamma)$, as captured by (10), with respect to γ yields after tedious, yet straightforward, simplifications:

(a.11) $$\frac{\partial f\left(\mu, \sigma^2, \rho, \gamma\right)}{\partial \gamma} = \frac{\beta'}{\gamma(\beta-1)^2} - \frac{(\gamma-1)\rho + \gamma(\rho - \gamma\mu)}{2(\rho - \mu_\pi)^2} < 0$$

where the inequality follows from $\beta' < 0$ and from $\rho > \mu_\pi > \gamma\mu$. □